while

$$\tau_2^1=\frac{RC_2}{1-K},\quad \tau_1^2=RC_1,$$

$$\tau_3^2=RC_3,\quad \text{and}\quad \tau_2^3=RC_2.$$

Thus the only nonzero coefficients are β_{12}, β_{23}, β_1, β_2, and β_3. Any one of these coefficients can be taken as unity, and the other four expressed in terms of it. The resulting transfer function can be then shown to have the form (after cancelling the common s factor)

$$H(s)=\frac{K\frac{C_3}{C_2}}{R(C_1+C_3)s+\left[1+\frac{C_3}{C_2}+(1-K)\frac{C_1}{C_2}\right]}. \tag{10}$$

Conclusion

A method has been presented for the analysis of active RC networks which is modular, effective, and intuitive to apply. Complex impedance calculations are unnecessary, and the final form of the transfer function is obtained immediately. A computation graph was developed which permits swift ordering of the time constant/pole calculations, thus permitting the method to be easily applied when the network is degenerate. The decomposition technique should prove to be a useful method for the analysis of practical active RC networks.

References

[1] R. D. Thornton *et al., Multistage Transistor Circuits*. New York: Wiley, 1965, ch. 1.
[2] B. L. Cochrun and A. Grabel, "A method for the determination of the transfer function of electronic circuits," *IEEE Trans. Circuits Syst.*, vol. CT-20, Jan. 1973.
[3] A. M. Davis and E. A. Moustakas, "Singular pole analysis of active filters," in *Proc. Fifteenth Allerton Conf. on Communication, Control, and Computing*, pp. 632–637, Sept. 1977.
[4] R. P. Sallen and E. L. Key, "A practical method of designing RC active filters," *IRE Trans. Circuit Theory*, vol. CT-2, May, 1955.
[5] A. Budak, *Passive and Active Network Analysis and Synthesis*. Boston, MA: Houghton-Mifflin, 1974.

A Generalized Algorithm for the Inversion of Cauer Type Continued Fractions

R. PARTHASARATHY AND SARASU JOHN

***Abstract*—A new generalized algorithm, which can be programmed on a digital computer, is established for performing the inversion of the Cauer type continued fractions.**

I. Introduction

The inversion of a continued fraction to a rational transfer function is of considerable practical interest in the area of circuits and systems [13], [14]. The use of continued fraction allows the construction of functional approximations to a given function without unwieldy calculations; this finds application in system reduction [10]; [11].

Manuscript received October 9, 1978; revised June 6, 1979.
R. Parthasarathy is with the Department of Electrical Engineering, Indian Institute of Technology, Madras, India.
S. John is with the Department of Electrical Engineering, Indian Institute of Technology, Madras, India, on leave from the Karnataka Regional Engineering College, Suratkal, India.

TABLE I

Type	a_i		b_i	
	i, odd	i, even	i, odd	i, even
Cauer I	$h_i s$	h_i	1	1
Cauer II	h_i'	$\frac{h_i'}{s}$	1	1
Modified Cauer	k_i	k_i	s	1

The first solution for the inversion problem was proposed by Chen and Shieh [1] and this was followed by the procedures in [2] and [3]. All the three procedures involve tedious computations. In [4]–[8] algorithms based on the Routh array for the inversion of Cauer I or Cauer II forms have been developed. These inversion algorithms start with the last quotient and successive quotients are added in the reverse order.

In contrast, the algorithm presented in this letter begins with the first quotient and progresses in the forward direction. It can be terminated at any desired point and a number of approximations of different orders are directly available from the rows of the inversion table.

II. Three Cauer Forms of Continued Fraction

Consider the following rational transfer function:

$$g(s)=\frac{q_1+q_2s+\cdots+q_ns^{n-1}}{p_1+p_2s+\cdots+p_{n+1}s^n} \tag{1}$$

where p_i's and q_i's are constants. Equation (1) can be expanded into the following generalized form of continued fraction:

$$g(s)=\cfrac{1}{a_1+\cfrac{b_1}{a_2+\cfrac{b_2}{a_3+\cfrac{b_3}{\ddots}}}} \tag{2}$$

where a_i's and b_i's are defined as in Table I.

The Cauer I and Cauer II forms, which are well known in the literature [10], give, respectively, a satisfactory approximation in the transient portion and in the steady-state portion of the system response. To obtain a good approximation to both the initial and steady-state portions of the response, Chuang [9] carried out the expansion about $s=0$ and $s=\infty$ alternately resulting in the following representation, which we call the modified Cauer form [15]:

$$g(s)=\cfrac{1}{k_1+\cfrac{s}{k_2+\cfrac{1}{k_3+\cfrac{s}{\ddots}}}}. \tag{3}$$

0098-4094/80/0500-0419$00.75 ©1980 IEEE

III. Formulation of the Algorithm

Let the continued fraction representation (2), employing quotients from a_i to a_j be denoted by $g_{i,j}(s)$, i.e.,

$$g_{i,j}(s)=\frac{q_{i,j}(s)}{p_{i,j}(s)} \tag{4}$$

$$=\cfrac{1}{a_i+\cfrac{b_i}{a_{i+1}+\ddots+\cfrac{b_{j-1}}{a_j}}}. \tag{5}$$

Then, obviously,

$$g_{i,j}(s)=\frac{1}{a_i+b_ig_{i+1,j}(s)}. \tag{6}$$

From (6)

$$\frac{q_{i,j}(s)}{p_{i,j}(s)}=\frac{p_{i+1,j}(s)}{a_ip_{i+1,j}(s)+b_iq_{i+1,j}(s)}. \tag{7}$$

This can be put in the form, analogous to that in [12]

$$\begin{bmatrix} p_{i,j}(s) \\ q_{i,j}(s) \end{bmatrix}=[M_i]\begin{bmatrix} p_{i+1,j}(s) \\ q_{i+1,j}(s) \end{bmatrix} \tag{8}$$

where

$$[M_i]=\begin{bmatrix} a_i & b_i \\ 1 & 0 \end{bmatrix}. \tag{9}$$

Now $g_{1,m}(s)$ represents the transfer function of a continued fraction with 'm' quotients from $i=1$ to $j=m$. From (8), we obtain

$$\begin{bmatrix} p_{1,m}(s) \\ q_{1,m}(s) \end{bmatrix}=[M^{(m)}]\begin{bmatrix} p_{m+1,m}(s) \\ q_{m+1,m}(s) \end{bmatrix} \tag{10}$$

where

$$[M^{(m)}]=[M_1]\cdots[M_m]=\begin{bmatrix} a^{(m)} & b^{(b)} \\ c^{(m)} & d^{(m)} \end{bmatrix}. \tag{11}$$

When the continued fraction is truncated after 'm' quotients, that is, $g_{m+1,m}(s)=0$, we have

$$g_{1,m}(s)=\frac{q_{1,m}(s)}{p_{1,m}(s)}=\frac{c^{(m)}(s)}{a^{(m)}(s)}. \tag{12}$$

IV. The Algorithm

A. Cauer I

The entries of $[M^{(m)}]$ defined in (11) for $m=1,2,\cdots$ can be evaluated recursively using (9) and (11); e.g., $[M^{(3)}]$ can be written as

$$[M^{(3)}]=\begin{bmatrix} h_1h_2h_3s^2+(h_3+h_1)s & h_1h_2s+1 \\ h_2h_3s+1 & h_2 \end{bmatrix} \tag{13}$$

and so on. Thus it will be seen that the entries 'a' and 'c' of the matrices $M^{(1)}, M^{(2)},\cdots$ possess the format given in Table II.

TABLE II

'a'-rows				'c'-rows		
1				0		
a_{11}	0			1		
a_{21}	1			c_{21}		
a_{31}	a_{32}	0		c_{31}	1	
a_{41}	a_{42}	1		c_{41}	c_{42}	
a_{51}	a_{52}	a_{53}	0	c_{51}	c_{52}	1
a_{61}	a_{62}	a_{63}	1	c_{61}	c_{62}	c_{63}
			⋮		⋮	

We start with the entries of $[M^{(1)}]$ built out of the first quotient to form the first two rows as $a_{01}=1$, $a_{11}=h_1$, $c_{01}=0$, and $c_{11}=1$. The remaining rows are formed using the following relations for a and c rows:

$$\begin{array}{ll} a_{i,j}=a_{i-1,j}h_i+a_{i-2,j-1} & c_{i,j}=c_{i-1,j}h_i+c_{i-2,j-1} \\ a_{i,1}=a_{i-1,1}h_i & c_{i,1}=c_{i-1,1}h_i \\ \text{for } i=2,3,\cdots & \text{for } i=2,3,\cdots \\ j=2,3,\cdots \dfrac{i}{2}+1 \text{ for } i \text{ even} & j=2,3,\cdots \dfrac{i}{2} \text{ for } i \text{ even} \\ j=2,3,\cdots \dfrac{i+3}{2} \text{ for } i \text{ odd} & j=2,3,\cdots \dfrac{i+1}{2} \text{ for } i \text{ odd.} \end{array} \tag{14}$$

Once the table is formed, the transfer function corresponding to the continued fraction with m quotients, $m=1,2,\cdots$ can be directly written from the entries in the $(m+1)$th row as

$$g_{1,m}(s)=\frac{\sum_{j=1}^{n} c_{m,j}s^{n-j}}{\sum_{j=1}^{n+1} a_{m,j}s^{n-j+1}} \tag{15}$$

where $n=m/2$ for m even and $(m+1)/2$ for m odd. It follows by inspection that

$$a_{m,n+1}=0 \text{ or } 1, \qquad \text{for } m \text{ odd or even}$$

and

$$c_{m,n}=1, \qquad \text{for } m \text{ odd.} \tag{16}$$

By substituting $c_{m,i}=q_{n-i+1}$ and $a_{m,i}=p_{n-i+2}$, we can reduce (15) to the form of (1).

B. Cauer II

It can be easily seen that the inversion array defined in the M-table is equally applicable to the Cauer II form. Thus the transfer function for the Cauer II form can be derived from (15) as

$$g_{1,m}(s)=\frac{\sum_{j=1}^{n} c_{m,j}s^{j-1}}{\sum_{j=1}^{n+1} a_{m,j}s^{j-1}}. \tag{17}$$

With the substitution $c_{m,i}=q_i$ and $a_{m,i}=p_i$, (17) can be brought to the form in (1).

C. Modified Cauer Form

For performing the inversion of the modified Cauer continued fraction, the M-table is constructed with the following modifications. The first two rows are the same as those in Table II where $a_{11}=k_1$. The subsequent rows are obtained through the recursive relations:

$$a_{i,j}=a_{i-1,j}k_i+a_{i-2,j-1} \qquad c_{i,j}=c_{i-1,j}k_i+c_{i-2,j-1}$$
$$a_{i,1}=a_{i-1,1}k_i \qquad c_{i,1}=c_{i-1,1}k_i$$
$$\text{for } i=2,4,\cdots \qquad \text{for } i=2,4,\cdots$$
$$j=2,3,\cdots,\frac{i}{2}+1 \qquad j=2,3,\cdots,\frac{i}{2}$$

and

$$a_{i,j}=a_{i-1,j}k_i+a_{i-2,j} \qquad c_{i,j}=c_{i-1,j}k_i+c_{i-2,j}$$
$$\text{for } i=3,5,\cdots \qquad \text{for } i=3,5,\cdots$$
$$j=1,2,\cdots,\frac{i+3}{2} \qquad j=1,2,\cdots,\frac{i+1}{2}. \tag{18}$$

From the M-table $g_{1,m}(s)$ can be written as

$$g_{1,m}(s)=\frac{\sum_{j=1}^{n} c_{m,j}s^{j-1}}{\sum_{j=1}^{n+1} a_{m,j}s^{j-1}}. \tag{19}$$

It will be noted that

$$a_{m,n+1}=0 \text{ or } 1, \qquad \text{for } m \text{ odd or even}$$

and

$$c_{m,n}=k_2, \qquad \text{for } m \text{ even}. \tag{20}$$

V. Conclusion

A new algorithm for inverting the Cauer I and Cauer II and modified Cauer forms of continued fraction is presented. Compared with the existing methods [4], [5], [8], [15], the proposed algorithm is superior computationally, when we bear in mind that a number of functions of different orders are simultaneously generated as the algorithm progresses.

References

[1] C. F. Chen and L. S. Shieh, "Continued fraction inversion by Routh's algorithm" *IEEE Trans. Circuit Theory*, vol. CT-16, pp. 197–202, May 1969.

[2] C. T. Chen, "A formula and an algorithm for continued fraction inversion," *Proc. IEEE*, vol. 57, pp. 1780–1781, Oct. 1969.

[3] L. S. Shieh, W. P. Schnider, and D. R. Williams, "A chain of factored matrices for Routh array inversion and continued fraction inversion," *Int. J. Contr.*, vol. 13, no. 4, pp. 691–703, 1971.

[4] K. S. Chao, I. P. S. Madan, and K. S. Lu, "Digital filter design by the continued fraction expansion," in *Proc. 16th Midwest Symp. on Circuit Theory*, Section VIII, pp. 3.1–3.7, Apr. 1973; [5] "On the inversion table of continued fraction expansion," *IEEE Trans. Automat. Contr.*, vol. AC-20, pp. 136–137, Feb. 1975.

[6] S. Vittal Rao and S. S. Lamba, "A note on continued fraction inversion by Routh's algorithm," *IEEE Trans. Automat. Contr.*, vol. AC-19, pp. 283–284, June 1974.

[7] R. Parthasarathy and H. Singh, "On continued fraction inversion by Routh's algorithm," *IEEE Trans. Automat. Contr.*, vol. AC-20, pp. 278–279, Apr. 1975.

[8] ——, "On continued fraction inversion by Routh's algorithm," *IEEE Trans. Automat. Contr.*, vol. AC-21, pp. 394, June 1976.

[9] S. C. Chuang, "Application of continued fraction method for modelling transfer functions to give more accurate initial transient response," *Electron. Lett.*, vol. 6, pp. 861–863, 1970.

[10] L. S. Shieh and M. J. Goldman, "Continued fraction expansion and inversion of the Cauer third form," *IEEE Trans. Circuit Syst.*, vol. CAS-21, pp. 341–345, May 1974.

[11] J. Schwartz and T. Edgar, "The development of simplified adaptive control models using the continued fraction expansion," *JACC, preprint*, pp. 484–494, 1974.

[12] C. F. Chen and W. T. Chang, "A matrix method for continued fraction inversion," *Proc. IEEE*, vol. 62, pp. 636–637, May 1974.

[13] Y. Shamash, "Continued fraction methods for the reduction of linear time-invariant systems," *IEEE Conf. on Computer Aided Control System Design*, Cambridge University, England, pp. 220–227, Apr. 1973.

[14] R. Parthasarathy, "System realization and identification using state-space technique," Ph.D. dissertation, ch. IV, Univ. of Roorkee, Roorkee, India, 1975.

[15] R. Parthasarathy and S. John, "System reduction using Cauer continued fraction expansion about $s=0$ and $s=\infty$ alternately," *Electron. Lett.*, vol. 14, no. 8, pp. 261–262, Apr. 1978.

A Radial Exploration Algorithm for the Statistical Analysis of Linear Circuits

KIRPAL S. TAHIM AND ROBERT SPENCE, FELLOW, IEEE

***Abstract*—This communication presents a new approach to the statistical analysis of linear circuits. The method is based on a two-stage sampling scheme in which random directions are first generated in multi-parameter component space, and then sample points are selected in each direction, the points being generated according to a modified probability density function within the tolerance region. An efficient tracking-sensitivity algorithm based on a matrix series expansion is utilized to approximate the circuit response values at the sample points. A technique for reducing the variance associated with the yield estimate is also discussed. The results are compared with those obtained by conventional Monte Carlo methods for a test example. Considerable savings in computational effort have been observed.**

I. Introduction

Manufacturing yield is defined as the proportion of manufactured circuits which meet the performance specifications, and has become an important parameter in computer-aided circuit design. Traditionally, straightforward Monte Carlo methods have been employed for yield estimation: circuits are simulated, analyzed, and tested against the specifications, and the proportion satisfying the specifications is taken to be an unbiassed estimate of the manufacturing yield. While intuitively simple to implement, this approach can, nevertheless, be computationally expensive and often prohibitively so. Recently, several attempts have been made to seek a less expensive approach. Initially, small-change sensitivity methods were used to estimate circuit responses at the sample points (1,2). To overcome approximation errors, large-change sensitivity methods with systematic exploration of a regionalised tolerance space were developed (3,4), but suffered from dimensional dependence. Other approaches (5,6) attempt to obtain a direct approximation to the Region of Acceptability. Quantile arithmetic operations have also been employed to compute the response probability density function (7). A recent development is aimed towards a reduction in the variance associated with the yield estimate by using importance sampling techniques (8,9).

Manuscript received December 28, 1977; revised June 13, 1979. This work was supported by the U.K. Science Research Council.

K. S. Tahim was with the Department of Electrical Engineering, Imperial College, London, England. He is now with the British Post Office, London, England.

R. Spence is with the Department of Electrical Engineering, Imperial College, London, England.

0098-4094/80/0500-0421$00.75 ©1980 IEEE