# High-Order Method for a Singularly Perturbed Second-Order Ordinary Differential Equation with Discontinuous Source Term Subject to Mixed Type Boundary Conditions

R. Mythili Priyadharshini[1] and N. Ramanujam[2]

[1] Department of Mathematical and Computational Sciences,
National Institute of Technology, Karnataka, India
[2] Department of Mathematics,
Bharathidasan University, Tamilnadu, India
http://www.bdu.ac.in/depa/science/ramanujam.htm

**Abstract.** In this paper, a singularly perturbed second-order ordinary differential equation with discontinuous source term subject to mixed type boundary conditions is considered. A robust-layer-resolving numerical method of high-order is suggested. An $\varepsilon$-uniform error estimates for the numerical solution and also to the numerical derivative are derived. Numerical results are presented, which are in agreement with the theoretical results.

**Keywords:** Singular perturbation problem, mixed type boundary conditions, discontinuous source term, Shishkin mesh, discrete derivative.

## 1 Introduction

Motivated by the works given in [2] - [5], the present paper considers the following singularly perturbed mixed type boundary value problem for second-order ordinary differential equation with discontinuous source term:

$$Lu \equiv \varepsilon u'' + a(x)u' = f(x), \quad \text{for all} \quad x \in \Omega^- \cup \Omega^+ \tag{1}$$

$$B_0 u(0) \equiv \beta_1 u(0) - \varepsilon\beta_2 u'(0) = A, \quad B_1 u(1) \equiv \gamma_1 u(1) + \gamma_2 u'(1) = B, \tag{2}$$

where $a(x) \geq \alpha > 0$, for $x \in \bar{\Omega}$, $|[f](d)| \leq C$, $\beta_1$, $\beta_2 \geq 0$, $\beta_1 + \varepsilon\beta_2 \geq 1$, $\gamma_1 > 0$, $\gamma_1 - \gamma_2 \geq 1$, the constants $A$, $B$, $\beta_1$, $\beta_2$, $\gamma_1$ and $\gamma_2$ are given and $0 < \varepsilon \ll 1$. We assume that $a(x)$ is sufficiently smooth function on $\bar{\Omega}$ and $f(x)$ is sufficiently smooth on $\Omega^- \cup \Omega^+$; $f$ and its derivatives have jump discontinuity at $x = d$. Since $f$ is discontinuous at $x = d$ the solution $u$ of (1), (2) does not necessarily have a continuous second derivative at the point d. Thus, $u(x)$ need not belong to the class of functions $C^2(\Omega)$ and $u \in Y \equiv C^1(\bar{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$. The novel aspect of the problem under consideration is that we take a source term in the differential equation which has a jump discontinuity at one or more points in the interior of the domain. This gives raise to a weak interior layer in the

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exact solution of the problem, in addition to the boundary layer at the outflow boundary point. In this paper, we constructed hybrid difference scheme (central finite difference scheme in the fine mesh region and mid-point difference scheme) for the BVP (1), (2) on Shishkin type meshes.

**Note:** Through out this paper, C denotes a generic constant that is independent of the parameter $\varepsilon$ and the dimension of the discrete problem $N$. Let $u : D \to \mathbb{R}$, $(D \subset \mathbb{R})$. An appropriate norm for studying the convergence of numerical solution to the exact solution of a singular perturbation problem is the maximum norm $\| u \|_D = \max_{x \in \overline{D}} |u(x)|$, [1]. We assume that $\varepsilon \leq CN^{-1}$ is generally the case for discretization of convection-diffusion equations.

**Theorem 1.** *[5] The problem (1), (2) has a solution* $u \in C^1(\bar{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$.

**Theorem 2.** *(Minimum Principle) [7] Let L be the differential operator in (1) and* $u \in Y$. *If* $B_0u(0) \geq 0,\ B_1u(1) \geq 0,\ Lu(x) \leq 0,$ *for all* $x \in \Omega^- \cup \Omega^+$ *and* $[u'](d) \leq 0,$ *then* $u(x) \geq 0,$ *for all* $x \in \bar{\Omega}$.

**Lemma 1.** *If* $u \in Y$ *then* $\| u \|_{\bar{\Omega}} \leq C \max\{|B_0u(0)|, |B_1u(1)|, \| Lu \|_{\Omega^- \cup \Omega^+}\}$.

## 2 Solution Decomposition and Mesh Discretization

The solution $u$ of (1), (2) can be decomposed into regular and singular components $u(x) = v(x) + w(x)$, where $v(x) = v_0(x) + \varepsilon v_1(x) + \varepsilon^2 v_2(x) + \varepsilon^3 v_3(x)$ and $v \in C^0(\Omega)$ is the solution of

$$Lv(x) = f(x),\ x \in \Omega^- \cup \Omega^+,\ B_0v(0) = B_0v_0(0) + \varepsilon B_0v_1(0) + \varepsilon^2 B_0v_2(0), \quad (3)$$
$$v(d) = v_0(d) + \varepsilon v_1(d) + \varepsilon^2 v_2(d), \quad B_1v(1) = B_1u(1). \quad (4)$$

Further, decompose $w(x) = w_0(x) + w_d(x)$, where $w_0 \in C^2(\Omega)$, is the solution of

$$Lw_0(x) = 0,\ x \in \Omega,\ B_0w_0(0) = B_0u(0) - B_0v(0), \quad B_1w_0(1) = 0, \quad (5)$$

and $w_d \in C^0(\Omega)$, is the interior layer function satisfying

$$Lw_d(x) = 0,\ x \in \Omega^- \cup \Omega^+,\ B_0w_d(0) = 0,\ [w_d'](d) = -[v'](d),\ B_1w_d(1) = 0. \quad (6)$$

**Lemma 2.** *[2,5] (Derivative Estimate) For each integer* $k$, *satisfying* $0 \leq k \leq 4$, *the derivatives of the solutions* $v(x)$, $w_0(x)$ *and* $w_d(x)$ *of (3),(4), (5) and (6) respectively, satisfy the following bounds*

$$\| v^{(k)} \|_{\Omega^- \cup \Omega^+} \leq C(1 + \varepsilon^{3-k}),\ |[v](d)|,\ |[v'](d)|,\ |[v''](d)| \leq C$$
$$|w_0^{(k)}(x)| \leq C\varepsilon^{-k} e^{-\alpha x/\varepsilon},\ x \in \bar{\Omega},\ |w_d(x)| \leq C\varepsilon,$$
$$|w_d^{(k)}(x)| \leq \begin{cases} C(\varepsilon^{1-k} e^{-\alpha x/\varepsilon}),\ x \in \Omega^- \\ C(\varepsilon^{1-k} e^{-\alpha(x-d)/\varepsilon}),\ x \in \Omega^+, \end{cases}$$

*where* $C$ *is a constant independent of* $\varepsilon$.