March 21, 2012

A STUDY OF HARMONIOUS AND COMPLETE COLORINGS OF DIGRAPHS

Thesis

Submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

by

LOLITA PRIYA CASTELINO

DEPARTMENT OF MATHEMATICAL AND COMPUTATIONAL SCIENCES
NATIONAL INSTITUTE OF TECHNOLOGY KARNATAKA,
SURATHKAL, MANGALORE - 575025
MARCH, 2012

Dedicated to my beloved parents and in-laws

DECLARATION

By the Ph.D. Research Scholar

I hereby declare that the Research Thesis entitled **A STUDY OF HARMO-NIOUS AND COMPLETE COLORINGS OF DIGRAPHS** which is being submitted to the **National Institute of Technology Karnataka, Surathkal** in partial fulfillment of the requirements for the award of the Degree of **Doctor of Philosophy** in **Mathematical and Computational Sciences** is a *bonafide report of the research work carried out by me.* The material contained in this Research Thesis has not been submitted to any University or Institution for the award of any degree.

MA08P03,   Lolita Priya Castelino

(Register Number, Name & Signature of the Research Scholar)

Department of Mathematical and Computational Sciences

Place: NITK, Surathkal.

Date: 19/03/2012.

CERTIFICATE

This is to *certify* that the Research Thesis entitled **A STUDY OF DIRECTED GRAPH LABELINGS** submitted by **SHIVARAJKUMAR**, (Register Number: MA08P02) as the record of the research work carried out by him, is *accepted as the Research Thesis submission* in partial fulfillment of the requirements for the award of degree of **Doctor of Philosophy**.

Dr. S. M. Hegde and Dr. Sudhakar Shetty

Research Guides

Dr. S. M. Hegde

Chairman - DRPC

Acknowledgements

I would like to acknowledge the debt of gratitude I owe to individuals and institutions without whose help this thesis would not have seen the light of day:

At the outset, I would like to express my profound gratitude to my supervisor **Prof. S. M. Hegde**, Professor, Department of MACS, National Institute of Technology, Karnataka, Surathkal, for suggesting the topic and problems and for his invaluable guidance, constant encouragement and deep involvement at every stage of my research. He had been very kind and patient to me throughout my studies. I deem it as my privilege to work under his able guidance. I ever remain grateful to him.

I am also thankful to my research co-guide **Dr. Sudhakar Shetty**, Professor, Department of Mathematics, Alva's Institute of Engineering & Technology, Moodubidri, for his help and support throughout my research. I owe a special debt of gratitude for his insightful suggestions and encouragement.

My sincere thanks to the members of Research Progress Assessment Committee (RPAC), **Dr. Prakash Raghavendra**, Department of Information Technology, National Institute of Technology Karnataka Surathkal, and **Dr. Santhosh George**, Department of MACS, National Institute of Technology Karnataka Surathkal, for taking their precious time to attend my seminars, to assess my progress and for their valuable suggestions and encouragement.

I acknowledge the financial support I received from the Department of Science and Technology, Government of India, for doing my Ph.D.

I am deeply indebted to my uncles and aunts, who have been pillars of strength, particularly **Mr. Paul** and **Mrs. Monthi Quadras** who have inspired me and have always been a guiding force right through my life.

At this juncture I think of my Parents, with profound sense of gratitude whose selfless sacrificial life and great efforts with pains to bring up my career.

I acknowledge with gratitude, the blessings of my beloved in-laws whose unceasing prayers have enabled me to reach this stage, with the gentle breeze of calmness, aura of peace and sweet scent of happiness.

I ever remember the good wishes, inspiration and whole hearted support in every ways received from my Sister, Brother, Brothers in-law and Co- sister, who have been the pillars of strength and guiding force to my life.

Finally, this thesis would not have been possible without the confidence, love and support of my beloved husband **Mr. Naval Castelino**. His loving concern, care and understanding made my task much easier.

Finally, I thank and remember each and every person who helped me directly or indirectly in carrying out this work.

And this is the Lord's doing. His grace has seen me through.

Place: NITK, Surathkal   Lolita Priya Castelino

Date: March, 2012

ABSTRACT OF THE THESIS

In this research work, we have extended the concept of harmonious colorings, complete colorings and set colorings of graphs to directed graphs.

A ***harmonious coloring*** of any digraph D is an assignment of colors to the vertices of D and the color of an arc is defined to be the ordered pair of colors to its end vertices such that all arc colors are distinct. The ***proper harmonious coloring number*** is the least number of colors needed in such a coloring. Also, we obtain a lower bound for the proper harmonious coloring of any digraphs and regular digraphs and investigate the proper harmonious coloring number of some classes of digraphs.

A ***complete coloring*** of a digraph D is a proper vertex coloring of D such that, for any ordered pair of colors, there is at least one arc of D whose endpoints are colored with this pair of colors. The ***achromatic number*** of D is the maximum number of colors in a proper complete coloring of D. We obtain an upper bound for the achromatic number of digraphs. Also, we find the achromatic number of some classes of digraphs.

We have extended the concept of set colorings of graphs to set colorings of digraphs. We have given some necessary conditions for a digraph to admit a strong set coloring (proper set coloring). We will characterize strongly (properly) set colorable digraphs. Also, we find the construction of strongly (properly) set colorable directed caterpillars.

Keywords: Harmonious colorings, proper harmonious coloring number, complete colorings, achromatic number, strong (proper) set coloring, digraphs.

Table of Contents

List of Figures

Chapter 1

Introduction

From Euler's solution to Konigsberg's Seven-Bridge problem (Euler 1741) to the present world wide web, graphs have been emerged as strong mathematical tools in many applications. The definition of a graph is very simple, but the theory developed based on it, is extremely vast. Some of the important topics of interest in graph theory include algebraic graph theory, coloring, domination, labeling, extremal graph theory and many more.

1.1 Preliminaries

In this section we record the definitions and results that are required for our study. For graph theoretic terminology, we refer (Harary 1972), (West 2003) and (Chartrand and L.Lesniak 2004).

A **graph** G is a finite non-empty set of objects called **vertices** together with a set of unordered pairs of distinct vertices of G, called **edges**. The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$ respectively. A graph H is called a **subgraph** of a graph G, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $e = uv$ is an edge of G, we say that u and v are adjacent and that each vertex is incident with e. The **degree** of a vertex v in graph G, denoted by $deg\ v$, is the number of edges incident with v. A graph G is called r- **regular** if $deg\ v = r$ for each $v \in V(G)$. A vertex

v is called a ***pendant vertex*** if $deg\ v = 1$ and an ***isolated vertex*** if $deg\ v = 0$. An edge e in a graph G is called a ***pendant edge*** if it is incident with a pendant vertex. A subset S of V is called an ***independent set*** of G of no two vertices of S are adjacent in G. The largest number of vertices in such a set is called the ***point independence number*** of G. An ***independent set of lines*** of G has no two of its lines adjacent and the maximum cardinality of such a set is the ***line independence number***. A collection of independent lines of a graph G is called a ***matching*** of G since it establishes a pairing of the vertices incident to them. $|V(G)| = n$ is called the ***order*** of G and $|E(G)| = m$ is called the ***size*** of G. A graph of order n and size m is called a (n, m) graph. The sum of the degrees of the vertices of a graph G is twice the number of edges, that is, $\sum deg\ v = 2m$, where the summation is taken over all vertices v of G.

An edge with identical ends is called a ***loop*** and an edge with distinct ends, a ***link***. A graph is ***simple*** if it has no loops and no two of its links join the same pair of vertices. A simple graph in which each pair of distinct vertices is joined by an edge is called a ***complete graph***. An ***empty graph*** is one with no edges. The ***complement*** $\overline{G}$ of a graph G also has $V(G)$ as its vertex set, but two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

A ***bipartite graph*** is one whose vertex set can be partitioned into two subsets X and Y, so that each edge has one end in X and the other end in Y. Such a partition (X, Y) is called a bipartition of the graph. A ***complete bipartite graph*** is a simple bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y. If $|X| = a$ and $|Y| = b$, such a graph is denoted by $K_{a,b}$. The graph that can be drawn in the plane without crossing the edges is known as ***planar graph***.

A ***digraph*** D consists of a finite set V of vertices and a collection of ordered pairs of distinct vertices. Any such pair (u, v) is called an ***arc*** of D. The arc (u, v) goes from u to v and is ***incident*** with u and v. We also say that u is adjacent ***to*** v and v is adjacent ***from*** u. The ***outdegree*** $od(v)$ of a vertex v is the number

of vertices adjacent from it and the ***indegree*** $id(v)$ of a vertex v is the number of vertices adjacent to it. A ***source*** in D is a vertex with indegree 0; a ***sink*** is the vertex with outdegree 0.

A digraph D is called ***symmetric*** if, whenever (u, v) is an arc of D, then (v, u) is also an arc of D. A digraph D is called an ***asymmetric digraph*** or an ***oriented graph*** if whenever (u, v) is an arc of D, then $(v, u)$is not an arc of D. A digraph D is ***complete*** if for every two distinct vertices u and v of D, at least one of the arcs (u, v) and (v, u) is present in D. The ***complete symmetric digraph*** of order n has both arcs (u, v) and (v, u) for every two distinct vertices u and v. A complete asymmetric digraph is called a ***tournament***. The ***underlying graph*** of a digraph D is that graph obtained by replacing each arc (u, v) or symmetric pairs (u, v), (v, u) of arcs by the edge uv. A digraph D is called ***regular of degree r*** or ***r-regular*** if $od\ v = id\ v = r$ for every vertex v of D.

A ***(directed) walk*** in a digraph is an alternating sequence of vertices and arcs, $v_0, x_1, v_1, ..., x_n, v_n$ in which each arc x_i is (v_{i-1}, v_i). The ***length*** of such a walk is the number of occurrences of arcs in it. A ***closed walk*** has the same first and last vertex and a ***spanning walk*** contains all the vertices. A ***trail*** is a walk in which all arcs are distinct and a ***path*** is a walk in which all vertices are distinct. A nontrivial closed trail of a digraph D is referred to as a ***circuit*** of D, and a circuit with all vertices distinct (except the first and the last) is called a ***cycle***. A ***semiwalk*** is again an an alternating sequence $v_0, x_1, v_1, ..., x_n, v_n$ of vertices and arcs, but each arc x_i may be either $(v_{i-1,i})$ or $(v_{i,i-1})$. A ***semipath***, ***semicycle***, and so forth, are defined as expected. An ***acyclic*** digraph contains no directed cycles. A digraph is called an ***out-tree*** if exactly one vertex has indegree 0 and all others have indegree 1. A digraph is called an ***in-tree*** if exactly one vertex has outdegree 0 and all others have outdegree 1.

An ***eulerian trail*** of a digraph D is an open trail of D containing all of the arcs and vertices of D, and an ***eulerian circuit*** is a circuit containing every arc

and vertex of D. A digraph that contains an eulerian circuit is called an ***eulerian digraph***.

Many combinatorial decision problems (those having a "yes" or "no" answer) are difficult to solve but once a solution is revealed, it is easy to verify the same. For example, the problem of determining whether a given graph G is k-colorable for some integer $k \geq 3$ is difficult to solve, that is, it is difficult to determine whether there exists a k-coloring of G. However, it is easy to verify that a given coloring of G is a k-coloring. It is only necessary to show that no more than k distinct colors are used and that adjacent vertices are assigned distinct colors. The collection of all such difficult-to-solve but easy-to-verify problems is denoted by ***NP***(Chartrand and Zhang 2009). The collection of all decision problems that can be solved in polynomial time is denoted by ***P***. The problems in the set NP have only one property in common with the problems belonging to the set P, namely: Given a solution to a problem in either set, the solution can be verified in polynomial time. Thus P⊆NP.

A problem is ***NP-hard*** if a polynomial-time algorithm for it could be used to construct a polynomial-time algorithm for each problem in NP. It is ***NP-complete*** if it belongs to NP and is NP-hard. The NP-complete problems are among the most difficult in the set NP and can be reduced from and to all other NP-complete problems in polynomial time.

1.2 Graphs and applications

1.2.1 Graph theory in computer science

- Trees and graphs as data structures.

- Inter connection networks in parallel computing.

- Inter connection networks in distributed computing.

- VLSI chips and PCBs.

- Optimal non-standard encoding of integers.

- Radar type codes.

- Self-orthogonal codes, a class of convolutional codes.

- Synchset codes.

- Missile guidance codes.

- The graph model.

1.2.2 Sociology

Suppose that the communication among a group of fourteen persons in a society is represented by the graph in Figure 1.1, where the vertices represent the persons and an edge represents the communication link between its two end vertices. Since the graph is connected, we know that all the members can be reached by any member, either directly or through some other members. But it is also important to note that the graph is a tree-minimally connected. The group cannot afford to lose any of the communication links.
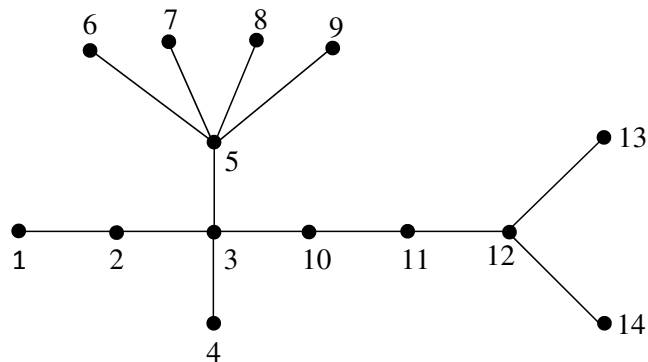
Figure 1.1: Communication link among 14 persons.

1.2.3 Chemistry

Given a chemical substance and some of its properties such as molecular weight, chemical composition, mass spectrum etc., the chemist would like to find out if this substance is a known compound. If he is able to identify this compound, he may like to know some additional properties of the compound, or if the compound is "new" he would like to know its structure, and then include it in the dictionary of known compounds. It is therefore essential to have a standard representation for a compound, and the representation must be compact, unambiguous, and amenable to classification.

In 1857, Arthur Cayley discovered trees while he was trying to count the number of structural isomers of the saturated hydrocarbons $C_k H_{2k+2}$. He used a connected graph to represent the $C_k H_{2k+2}$ molecule. Corresponding to their chemical valencies, a carbon atom was represented by a vertex of degree four and a hydrogen atom by a vertex of degree one. The total number of vertices in such a graph is $p = 3k + 2$ and the total number of edges is $q = 3k + 1$. Since the graph is connected and the number of edges is one less than the number of vertices, it is a tree. Thus the problem of counting structural isomers of a given hydrocarbon becomes the problem of counting trees. For example, the structural graph of aminoacetone $C_3 H_7 NO$ is shown in Figure 1.2. For compactness the hydrogen atoms are omitted as they are implied by every unused valence of the other atoms.
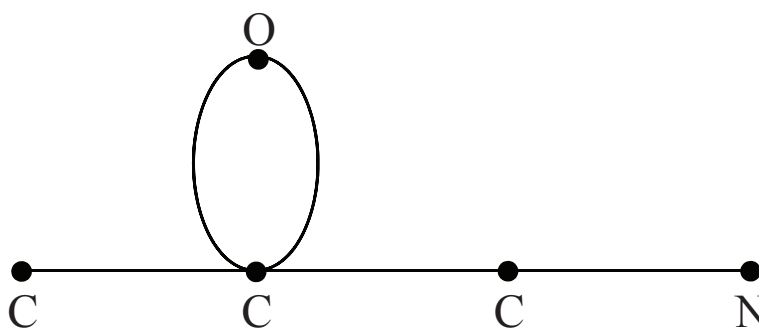
Figure 1.2: Structure of aminoacetone.

1.2.4 Graph theory in Operations Research

Graph theory is a very natural and powerful tool in Combinatorial Operations Research. The traveling salesman problem, finding the shortest spanning tree in a weighted graph, obtaining an optimal matching of jobs and men and locating the shortest path between two vertices in a graph are some examples of the uses of graph theory in operations research. One of the most popular and successful applications of networks in operations research is in the planning and scheduling of large complicated projects. The two best known names in this connection are Critical Path Method and Program Evaluation and Review Technique.

1.2.5 Ambiguities in X-ray Crystallography

Determination of crystal structures from X-ray diffraction data has long been a concern of crystallographers. These inherent ambiguities in the X-ray analysis of crystal structures have been studied by Patterson (1944), Garrido (1951) and Franklin (1974). Research into these ambiguities is concerned with determination of arrangements of set of points from a knowledge of the vector distances for these points. Extensive results for the case of an infinity of points arranged periodically have been achieved by Patterson and Garrido.

1.2.6 Communication network labeling

In a small communication network, it may be desirable to assign each user terminal a "node number", subject to the constraint that all the resulting edges receive distinct numbers. In this way, the numbers of any two communicating terminals automatically specify the link number of the connecting path; and conversely, the path number uniquely corresponds to the pair of user terminals which it interconnects. Properties of a potential numbering system for such networks have been explored under the guise

of gracefully numbered graphs. That is, the properties of graceful graphs provide design parameters for an appropriate communication network. For example, the maximum number of links in a network with m transmission centres can be shown to be asymptotically limited to not more than $\frac{2}{3}$ of all possible links when m is large.

1.2.7 Linguistics

Graphs have been used in linguistics to depict parsing diagrams. The vertices represent words and word strings and the edges represent certain syntactical relationships between them. A set of words (vocabulary) and a set of rules (grammar) for forming strings (sentences) characterize a language. The language then is a set of all legal strings so generated. One problem in computational linguistics is to identify whether or not a given string belongs to a language, whose vocabulary and grammar are given.

1.2.8 Social relations

In 1936, Lewin (1936) proposed that the "life space" of an individual can be represented by a planar map. In such a map, the regions would represent the various activities of a person, such as his work environment, his home and his hobbies. It was pointed out that Lewin was actually dealing with graphs. This viewpoint led the psychologist at the Research Centre for Group Dynamics to another psychological interpretation of a graph in which people are represented by vertices and interpersonal relations by edges. Such relations include love, hate, communication and power.

Graph theory has also been used in Economics, Logistics, Cybernetics, Artificial Intelligence, Pattern Recognition, Genetics, Reliability Theory, Fault Diagnosis in Computers and the study of Martian Canals.

1.3   Graph colorings

The Four Color Problem

Can the countries of every map be colored with four or fewer colors so that every two countries with a common boundary are colored differently?

While this problem may seem nothing more than a curiosity, it is precisely this problem that would prove to intrigue so many for so long and whose attempted solutions would contribute so significantly to the development of the area of Mathematics known as Graph Theory and especially to the subject of graph colorings.

Coloring the regions, vertices and edges of maps and planar graphs, inspired by the desire to solve the Four Color Problem, has progressed far beyond this - to coloring more general graphs and even to reinterpreting what is meant by coloring.

There is little doubt that the best known and most studied area within graph theory is coloring. With its origins embedded in attempts to solve the famous Four Color Problem, graph colorings has become a subject of great interest, largely because of its diverse theoretical results, its unsolved problems, and its numerous applications.

There are three types of colorings.

1. Vertex Colorings.

2. Region Colorings in the case of planar graphs.

3. Edge Colorings.

The problems in graph colorings that have received the most attention involve coloring the vertices of a graph. Furthermore, the problems in vertex colorings that have been studied most often are those referred to as proper vertex colorings.

A ***proper vertex coloring*** of a graph G is an assignment of colors to the vertices of G, one color to each vertex, so that adjacent vertices are colored differently. When it is understood that we are dealing with a proper vertex coloring, we ordinarily refer

to this more simply as a ***coloring*** of G. While the colors used can be elements of any set, actual colors (such as red, blue, green and yellow) are often chosen only when a small number of colors are being used; otherwise, positive integers (typically $1, 2, ..., k$ for some positive integer k) are commonly used for the colors. A reason for using positive integers as colors is that we are often interested in the number of colors being used. Thus, a (proper) coloring can be considered as a function $c : V(G) \to \mathbb{N}$ (where $\mathbb{N}$ is the set of positive integers) such that $c(u) \neq c(v)$ if u and v are adjacent in G. If each color used is one of k given colors, then we refer to the coloring as a k-***coloring***. In a k-coloring, we may then assume that it is the colors $1, 2, ..., k$ that are being used. A graph G is k-***colorable*** if there exists a k-coloring of G. The minimum positive integer k for which G is k-colorable is the ***chromatic number*** of G.

Let G be a plane graph. Then G is k-***region colorable*** if each region of G can be assigned one of k given colors so that neighboring (adjacent) regions are colored differently.

An ***edge coloring*** of a graph G is an assignment of colors to the edges of G, one color to each edge. If adjacent edges are assigned distinct colors, then the edge coloring is a ***proper edge coloring***. A proper edge coloring that uses colors from a set of k colors is a k-***edge coloring***.

There is no general formula for the chromatic number of a graph. Consequently, we will often be concerned and must be content with **(1) determining the chromatic number of some specific graphs of interest or of graphs belonging to some classes of interest and (2) determining upper and/or lower bounds for the chromatic number of a graph.**

1.4   Harmonious colorings of graphs

Definition 1.4.1. *A harmonious coloring (Hopcroft and Krishnamoorthy 1983) of a graph G is an assignment of colors to the vertices of G and the color of an edge is defined to be the unordered pair of colors to its end vertices such that all edge colors are distinct. The harmonious coloring number is the least number of colors in such a coloring and is denoted by $h(G)$.*

Example:
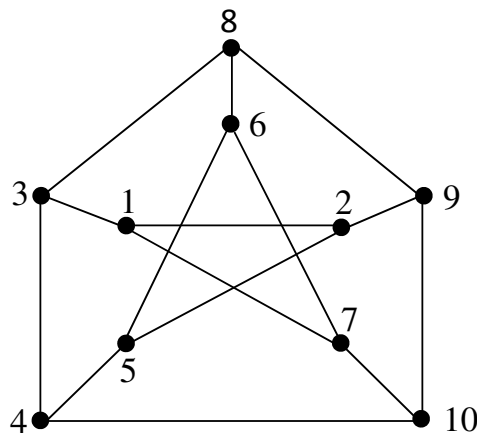
Figure 1.3: A harmonious coloring of Petersen Graph

1.4.1   A brief review of harmonious colorings

The first paper on harmonious graph coloring was published in 1982 by Frank et al. (1982). However, the proper definition of this notion is due to Hopcroft and Krishnamoorthy (1983) in 1983. In the same paper they have proved that determining the harmonious coloring number of a graph is NP-hard.

In general the harmonious coloring problem has been viewed as an Eulerian path decomposition of graphs. Whenever such a decomposition is possible, it is possible to find the harmonious coloring number of graphs.

The concept of harmonious coloring led to following two categories of problems:

(i) Finding the exact harmonious coloring number if it can be found.

The following results gives the classes of graphs for which the harmonious coloring number has been found by different authors.

- For any graph G of order n and diameter at most two, we have $h(G) = n$. Also, $h(G) = n$ for almost all graphs (Miller and Pritikin 1991).

- The harmonious coloring number of a path P_n on n vertices given by
$$h(P_n) = \begin{cases} 2k & if \ \binom{2k-1}{2} < n-1 \leq \binom{2k}{2} - (k-1), \\ 2k+1 & if \ \binom{2k}{2} - (k-1) < n-1 \leq \binom{2k+1}{2}. \end{cases}$$
in (Miller and Pritikin 1991).

The following result is another view of the harmonious coloring number of a path on n vertices and was given by Mitchem (1989).

- Let P_n be a path on n vertices. Let k be the smallest integer such that $\binom{k}{2} \geq n-1$. If k is odd or if k is even and $n-1 = \binom{k}{2} - j$, where $j = \frac{1}{2}(k-2), \frac{k}{2}, ..., k-2$, then $h(P_n) = k$. Otherwise $h(P_n) = k+1$.

Georges (1995) extended the above result to determine the harmonious coloring number of a collection of disjoint paths.

- Let $X = \{P^1, P^2, ..., P^n\}$ be a collection of nontrivial disjoint paths where P^i has a_i edges, $1 \leq i \leq n$. Let k be the smallest positive integer such that $\sum_{i=1}^{n} a_i \leq \binom{k}{2}$. If k is odd, then $h(X) = k$. If k is even and $\sum_{i=1}^{n} a_i \leq \binom{k}{2} - \frac{k}{2} + min\{n, \frac{k}{2}\}$, then $h(X) = k$, otherwise $h(X) = k+1$.

- Let bP_n denote the graph that is b disjoint copies of P_n. Let k be the smallest integer such that $b(n-1) \leq \binom{k}{2}$. If k is odd, then $h(bP_n) = k$. If k is even

and $2b(n-2) \leq k(k-2)$, then $h(bP_n) = k$, otherwise $h(bP_n) = k+1$ (Georges 1995).

The following result due to Mitchem (1989) is on harmonious coloring number of a tree of order n.

- Let k be the least integer such that $\binom{k}{2} \geq n-1$. Then for each t, $k \leq t \leq n$, there is a tree T of order n such that $h(T) = t$.

The following is the corollary of the above result.

- Let $n \leq m \leq \binom{n}{2}$. Let k be the smallest integer such that $\binom{k}{2} \geq m$. Then for any t, $k \leq t \leq n$, there exists a graph G with n vertices and m edges such that $h(G) = t$.

The harmonious coloring number of a cycle on n vertices is given in the following theorem.

- Let $r + s = n$ where $3 \leq r \leq s$ and let k be the least integer such that $\binom{k}{2} \geq n$. If k is odd and $n \neq \binom{k}{2} - i$ where $i = 1$ or 2, then $h(C_n) = h(C_r \cup C_s) = k$. If k is even and $n \neq \binom{k}{2} - i$ where $i = 0, 1, ..., \frac{k}{2} - 1$, then $h(C_n) = k$. Otherwise $h(C_n) = k + 1$ (Mitchem 1989).

The extension of the above result to the collection of disjoint cycles was given by Georges (1995).

- Let $X = \{C^1, C^2, ..., C^k\}$ be a collection of disjoint cycles where C^i has c_i edges, $1 \leq i \leq k$, and let n be the smallest integer such that $\sum_{i=1}^{k} c_i \leq \binom{n}{2}$. If n is odd and $n \geq 2k + 3$, then $h(X) = h(C_p)$, where $p = \sum_{i=1}^{n} c_i$

We know that for a complete graph K_n on n vertices, the harmonious coloring number is equal to the number of vertices, i.e. $h(K_n) = n$. The following theorem due to Georges (1995) gives the harmonious coloring number of a collection of disjoint complete graphs which have an arbitrary number of vertices.

- Let $X = \{K^1, K^2, ..., K^n\}$ be a collection of disjoint complete graphs where K^i has k_i vertices, $i = 1, 2, ..., n$. If $min_i\{k_i\} \geq n - 1$, then $h(X) = \sum_{i=1}^n k_i - \binom{n}{2}$.

The next result follows as an immediate corollary of the above theorem.

- Let bK_n denote the graph that consists of b disjoint copies of complete graphs on n vertices. Then for any $1 \leq b \leq n + 1$, it follows that $h(bP_n) = k$. If k is even and $2b(n - 2) \leq k(k - 2)$, then $h(bK_n) = bn - \binom{b}{2}$ (Georges 1995).

The harmonious coloring number of the disjoint union of two arbitrary complete bipartite graphs is given in (Georges 1995) as follows:

- Let r, s, p and q be positive integers with $r = max\{r, s, p, q\}$. Then
$$h(K_{r,s} \cup K_{p,q}) = \begin{cases} r + s & if \ \ r \geq p + q, \\ s + p + q & if \ \ r < p + q. \end{cases}$$

(ii) If finding the harmonious coloring number is hard or traceable, then we can compute upper and lower bounds and the results so obtained in this view are general in nature.

The following results have been obtained or improved by many authors.

- For any graph G with n vertices and maximum degree $\Delta(G) = \Delta$, $h(G) \leq (\Delta^2 + 1)\lceil \sqrt{n} \rceil$ (Lee and Mitchem 1987).

- Let bK_n denote the graph formed from b disjoint copies of the complete graph K_n. Then $h(bK_n) \leq (n^2 - 2n + 2)\lceil \sqrt{bn} \rceil$ (Lee and Mitchem 1987).

In 1991, Zhikang Lu provided an upperbound which is an improvement on the bound given by Lee and Mitchem. McDiarmid and Xinhua also improved on this bound independently in the same year. The following is due to Lu (1993).

- For any graph G with order n and maximum degree $\Delta(G) = \Delta$, $h(G) \leq 2\Delta\lfloor \sqrt{n} \rfloor$.

Let $\hat{\delta}$ be the maximum of the minimum degrees, taken over all induced subgraphs of a graph G. It is clear to see that $\hat{\delta} \leq \Delta$. McDiarmid and Xinhua (1991) proved that

- Let G be a graph with $\hat{\delta} \leq \alpha \leq \Delta$ and let $4\alpha\Delta \leq n - 1$. Then
  $h(G) \leq 2\sqrt{\alpha\Delta(n-1)} - \frac{7}{8}\alpha\Delta + (\Delta - \alpha)(\hat{\delta} - 1) + 1$.

We can use the above result to prove the following two theorems which gives two upper bounds for $h(G)$ (McDiarmid and Xinhua 1991).

- For any nontrivial graph G with n vertices, m edges and maximum degree $\Delta(G) = \Delta$, $h(G) \leq 2\Delta\sqrt{n-1}$.

- For any nontrivial graph G with n vertices, m edges and maximum degree $\Delta(G) = \Delta$,
  $$h(G) \leq \begin{cases} \frac{33}{16}\sqrt{\hat{\delta}\Delta(n-1)} & if \ \ \hat{\delta} > 8, \\ 2\sqrt{\hat{\delta}\Delta(n-1)} & \hat{\delta} \leq 8. \end{cases}$$

We can deduce the following result from the above theorem.

- If T is a nontrivial tree of order n, then $h(T) \leq 2\sqrt{\Delta n}$.

In 1994, Krasikov and Roditty (1994) improved the upper bound originally given by Lee and Mitchem and that has already been improved by Zhikang Lu as well as McDiarmid and Xinhua.

- For any graph G of order n, $h(G) \leq \lceil \frac{n}{t} \rceil$, where $t = max\{1, \lceil \frac{1}{4n}(\Delta + 1 + \sqrt{(\Delta+1)^2 + 8n}) \rceil - 1\}$. Moreover, there is a harmonious coloring of G with $\lceil \frac{n}{t} \rceil$ colors such that each color contains at most t vertices.

Edwards and McDiarmid (1994) gave following upper bound for the harmonious coloring number of any graph G.

- Let G be any graph with m edges and with maximum degree Δ. Then for any integer $k \geq 2$,
  $h(G) \leq max\{|V_{(k)}|, 2\sqrt{2m(k-1)} + (2k-3)\Delta\}$.

If $k = 2$ in the above result, we have the following corollary (Edwards and McDiarmid 1994).

- For any tree T with m edges, $h(T) \leq 2\sqrt{2m} + \Delta$.

Edwards (1998) proved the following result.

- Let G be a graph with n vertices and m edges. Let $k = \hat{\delta}(G)$. Let $r = \lceil \frac{\sqrt{8m(k+\Delta)+A^2}}{2} + \frac{A}{2} \rceil$, where $A = (2\Delta k + \Delta^2 - k^2)$. Then $h(G) \leq r$.

1.4.2 Applications of harmonious colorings of graphs

The harmonious coloring problem has several applications.

- Harmonious coloring problem has potential applications in communication networks (i.e. transportation networks, computer networks etc.), since requesting each edge to have a unique color that depends on the assignment of colors on the vertices can be translated as assigning codes to the network nodes such that each communication link can be distinguished.

- Kubale (2004) considers radio navigation systems (aviation guiding systems in bad weather conditions or in case of invisibility on ground objects) as a possible area for application of harmonious coloring number. The airway network system consists of several airways. We could place one radio beacon on each terminal point of each airway and an aircraft can determine its position by counting the frequencies of the two beacons. For safety reasons no two proximate beacons should be assigned the same frequency and no two airways can have radio beacons sending same signals on both end points. In order to minimize the number of frequencies that can be used in the beacons, we should consider an optimal harmonious coloring on the graph where nodes represent the positions of the radio beacons and edges represent the airways.

- Cichelli (1980) investigates the potential applications of harmonious coloring problem in data compression (design of minimal perfect hash functions).

1.5 Complete coloring of graphs and achromatic number

Definition 1.5.1. *A complete coloring of a graph G is a proper vertex coloring of G such that, for any pair of colors, there is at least one edge of G whose endpoints are colored with this pair of colors. The achromatic number of G, denoted $\psi(G)$, is the greatest number of colors in a complete coloring of G.*

Example:
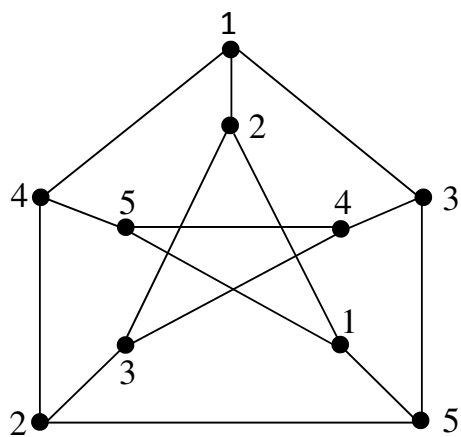
Figure 1.4: A complete coloring of Petersen Graph

1.5.1 A brief review of complete colorings and achromatic number

The achromatic number was introduced by Harary et al. (1967) in 1967. They considered homomorphisms from a graph G onto a complete graph K_n. A homomorphism from a graph G to a graph G' is a function $\phi : V(G) \to V(G')$ satisfying $u \sim v \Rightarrow \phi(u) \sim \phi(v), \; \forall u, v \in G$. This induces an obvious mapping from $E(G)$ to $E(G')$. It is easy to see that a complete coloring of G with n colors corresponds precisely to a complete homomorphism of G onto K_n. i.e. one whose induced edge mapping maps $E(G)$ onto $E(K_n)$. They considered the largest n for which such homomorphism exists. This was later named the achromatic number $\psi(G)$ by Harary and Hedetniemi (1970).

Determining the achromatic number of a graph is shown to be NP-hard for general graphs by Yannakakis and Gavril (1980) and it is further shown to be NP-hard when restricted to bipartite graphs (Farber et al. 1986), to the family that is the intersection of cographs and interval graphs (Bodlaender 1989) and even trees (Cairnie and Edwards 1997) and in 2007, NP-completeness was shown for the restricted cases of bipartite permutation and quasi-threshold graphs (Asdre and Nikolopoulos 2007).

The problem of determining the achromatic number on paths was given by Krysta and Lorys (2006). Let $q(m)$ be the greatest integer number l such that $m \geq \binom{l}{2}$ and $q(m) = \lfloor \frac{1+\sqrt{1+8m}}{2} \rfloor$.

- Let P_m be a path of m edges. If $q(m)$ is odd or $q(m)$ is even and $m \geq \frac{q(m)^2}{2} - 1$ then $\psi(P_m) = q(m)$ else $\psi(P_m) = q(m) - 1$.

The simplest upperbound for $\psi(G)$ is due to Harary and Hedetniemi (1970).

- For any graph G, $\psi(G) \leq \alpha_o(G) + 1$ where $\alpha_o(G)$ is the size of the minimum vertex cover of G.

The best possible upper and lower bounds for the achromatic number of the disjoint union $G \cup H$ of two graphs G and H is given by Hell and Miller (1992).

- For any two graphs G and H, $max\{\psi(G), \psi(H)\} \leq \psi(G \cup H) \leq \psi(G) \cdot \psi(H)$.

In the same paper Hell and Miller gave the best possible lower bound for the product of two graphs.

- For graphs G and H, $\psi(G \times H) \geq \psi(G) + \psi(H)$ unless (assuming without loss of generality that $\psi(H) \leq \psi(G)$)

  (i) $\psi(H) = 3$ and $\psi(G) \leq 5$, in which case $\psi(G \times H) \geq \psi(G) + \psi(H) - 1$.

  (ii) $\psi(H) = 2$, in which case $\psi(G \times H) \geq \psi(G) + \psi(H) - 2$.

For achromatic number, the best lower bound for trees is that of Farber et al. (1986) They show that extending a tree by a fairly small number of edges must increase the achromatic number and so obtain a recurrence which leads to an upper bound for the number of edges of a tree with maximum degree at most d and achromatic number at most k.

- Let T be a tree with m edges, maximum degree at most d and satisfying $\psi(T) \leq k$. Then
$$m \leq \begin{cases} (k-1)d + \binom{k-1}{2} & \text{if } k \leq d, \\ (k-1)k + \binom{d-1}{2} & \text{if } k \geq d. \end{cases}$$

- Let T be a tree with m edges and maximum degree Δ. Then $\lfloor \frac{3}{2} + (m - \binom{\Delta-1}{2} - \frac{3}{4})^{\frac{1}{2}} \rfloor \leq \psi(T) \leq q(m)$.

An approximation algorithm is one which delivers an approximate solution to a problem. For a maximization problem such as the determination of achromatic number, an algorithm has *approximation ratio* α if it always produces a solution whose value is at least $\frac{1}{\alpha}$ of the optimum.

Chaudhary and Vishwanathan (2001) gave a polynomial time approximation algorithm for the achromatic number with approximation ratio $O(n/\sqrt{\log n})$. For graphs of girth at least 7, they gave a simple algorithm with approximation ratio $O(n^{7/20})$.

1.5.2   Applications of complete colorings of graphs

Complete coloring problem can be directly applicable in network design, specifically in clustering (Halldorsson 2004). A desired property when building networks is that of having small diameter while the graph is as sparse as possible. Namely, we should like to have small diameter inside the clusters and as few edges as possible connecting different clusters while maintaining high connectivity. Consider an existing network that we would like to cluster. Obtain a maximum complete partition. The clusters can be obtained directly from the color classes. A large number of clusters implies a small number of machines inside each cluster. This makes it possible to add fast (but expensive) means of communication inside each cluster. Furthermore, since the coloring is complete, communication between clusters can be performed directly for every pair of clusters.

At this point it is worth noticing that both problems are hard even when restricted to many graph families in which NP-hard problems usually become tractable. There are a few families for which we can have exact solutions in polynomial time including paths, cycles, unions of paths and cycles, stars, complete graphs, complete bipartite graphs and threshold graphs.

It is interesting to notice that a harmonious coloring of a subgraph H of G cannot easily be extended to a coloring of G. On the contrary it might be the case that even introducing new colors is not enough to extend a coloring of H to a coloring of G. That is because we might have two vertices u and v that were assigned the same colors but are both neighbors of some vertex w in graph G. But a complete coloring with k colors of a graph H can be easily extended to a complete coloring with k colors of a supergraph G of H.

Example:
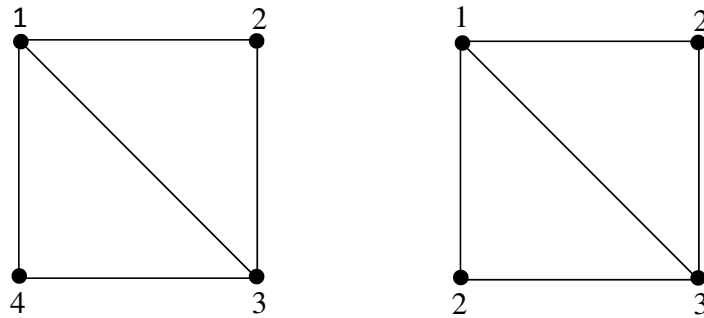
Figure 1.5: A harmonious and complete coloring of a graph

1.6   Set colorings of graphs

The notion of set coloring of a graph has been introduced by Hegde (2009) in 2009. Acharya (1983) has initiated a general study of labeling of the vertices and the edges of a graph using subsets of a set and indicated their potential application in a variety of other areas of human enquiry.

Let X be a nonempty set of colors, 2^X denote the set of all possible combinations of colors (or power set) of X and $Y(X) = 2^X \setminus \emptyset$. For any two subsets A and B of X, let $A \oplus B$ denote the symmetric difference of A, B and be given by $A \oplus B = (A \cup B) - (A \cap B)$.

Given a (p, q) graph $G = (V, E)$ and a nonempty set X of colors, we define a function f on the vertex set V of G as an assignment of subsets of X to the vertices of G, and given such a function f on the vertex set V we define $f^\oplus$ on the set of edges E as an assignment of the colors $f^\oplus(e) = f(u) \oplus f(v)$ to the edge $e = uv$ of G.

Let $f(G) = \{f(u) : u \in V\}$ and $f^\oplus(G) = \{f^\oplus(e) : e \in E\}$. We call f a ***set coloring*** of G if both $f(G)$ and $f^\oplus(G)$ are injective functions. A graph is called ***set colorable*** if it admits a set coloring.

A set coloring f of G is called a ***strong set coloring*** if $f(G)$ and $f^{\oplus}(G)$ are disjoint subsets of X and further, they form a partition of $Y(X)$. If G admits such a coloring then G is called a ***strongly set colorable*** graph.

A set coloring f is called a ***proper set coloring*** if $f^{\oplus}(G) = Y(X)$. If a graph G admits such a set coloring, then it is called a ***proper set colorable*** graph.

The set coloring number $\sigma(G)$ of a graph G is the least cardinality of a set X with respect to which G has a set coloring. Further if $f : V \to 2^X$ is a set coloring of G with $|X| = \sigma(G)$, we call f an ***optimal set coloring*** of G.

Figure 1.6 gives examples of (a) strongly, (b) properly, (c) non-strongly and non-properly set colorable graphs.
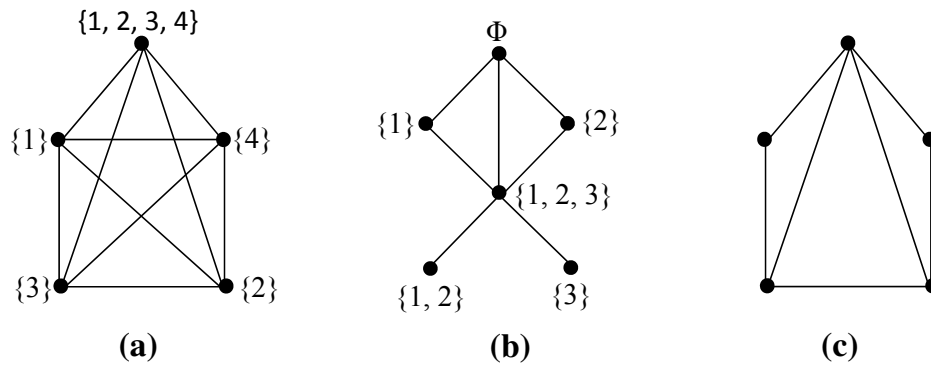
Figure 1.6: Stongly, Properly, Non-strongly and non-properly set colored graphs

Boutin et al. (2010) considered the problem on paths and complete binary trees, and showed that it can be reduced to the computation of a transversal in a special Latin square, i.e., the XOR table. Also they investigated a variation of the problem called strong set coloring and provided an exhaustive list of all graphs being strongly set colorable with at most 4 colors.

Balister et al. (2011) disproved a conjecture that the path $P_{2^{n-1}}$ is strongly set colorable for $n \geq 5$. Also they proved another conjecture of Hegde on a related type of set coloring of complete bipartite graphs.

1.7   Outline of the Thesis

In Chapter 2, we define harmonious colorings of digraphs. We obtain a lower bound for the proper harmonious coloring number and investigate the proper harmonious coloring number of some classes of digraphs.

In Chapter 3, we obtain a lower bound for the proper harmonious coloring number of r-regular digraphs and investigate the same for regular digraphs such as oriented torus and circulant digraph.

In Chapter 4, we define complete colorings of digraphs. We obtain an upper bound for the achromatic number of digraphs. Also we find the achromatic number of some classes of digraphs.

In Chapter 5, we define set coloring of digraphs. Some necessary conditions have been given for a digraph to admit a strong set coloring (proper set coloring). We characterize strongly (properly) set colorable digraphs such as directed stars, directed bistars etc. Also, we find the construction of strongly (properly) set colorable caterpillars.

In Chapter 6, we give a conclusion and scope for future research.

Chapter 2

Harmonious Colorings of Digraphs

A coloring of a graph can be described by a function that maps pieces of a graph (vertices - vertex coloring, edges - edge coloring or both) into some set of numbers (possibly $\mathbb{N}$, $\mathbb{Z}$ or even $\mathbb{R}$) usually called **colors**, such that some property is satisfied.

In this chapter we focus on a type of vertex coloring called harmonious coloring of directed graphs. Also, we obtain the lower bound for the harmonious coloring number of digraphs and investigate the same for different types of digraphs.

2.1 Introduction

The following is an extension of harmonious colorings to directed graphs.

Definition 2.1.1. *Let D be a directed graph with n vertices and m arcs. A function $f : V(D) \to \{1, 2, ..., k\}$, where $k \leq n$ is said to be a **harmonious coloring** of D if for any two arcs (x, y) and (u, v) of D, the ordered pair $(f(x), f(y)) \neq (f(u), f(v))$. If the pair (i, i) is not assigned, then f is called a **proper harmonious coloring** of D. The minimum k for which D admits a proper harmonious coloring is called the **proper harmonious coloring number** of D and is denoted by $\overrightarrow{\chi_h}(D)$.*

Example:
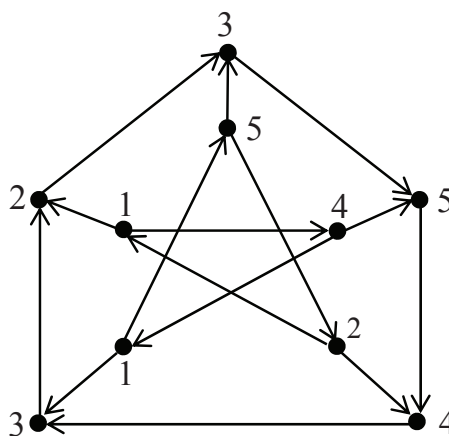
Figure 2.1: A proper harmonious coloring of oriented Petersen graph.

In the above figure a proper harmonious coloring of oriented Petersen graph is displayed.

There are two lower bounds for proper harmonious coloring number of D. Denote the maximum indegree or outdegree of any vertex v of D by Δ and the number of arcs of D by m.

(i) Any vertex and all of its neighbors must receive distinct colors, and thus $\overrightarrow{\chi_h}(D) \geq \Delta + 1$.

Also, the proper harmonious coloring number of D cannot be greater than the number of vertices of D. Let n be the number of vertices of D. Then $\overrightarrow{\chi_h}(D) \leq n$.

Thus, $\Delta + 1 \leq \overrightarrow{\chi_h}(D) \leq n$.

(ii) There must be at least as many pairs of colors as there are arcs. In a proper harmonious k- coloring, the number of possible pairs of colors is $k(k-1)$. Since the number of arcs, m in D must be less than or equal to the total number of possible pairs of colors, it follows that

$m \leq k(k-1)$

i.e., $m < k(k-1) + 1$

i.e., $k(k-1) \geq m$

i.e., $k^2 \geq m + k$

i.e., $k^2 > m$

i.e., $k > \sqrt{m}$.

In the next section, we improve the above lower bound.

2.2 Lower bound for the proper harmonious coloring number of digraphs

The following theorem gives a lower bound for the proper harmonious coloring number.

Theorem 2.2.1. *For any digraph D, $\overrightarrow{\chi_h}(D) \geq \lceil \frac{1+\sqrt{4m+1}}{2} \rceil$, where m is the number of arcs of D.*

Proof. Let D be a digraph. Then D is colored with k colors using proper harmonious coloring. Then the possible number of ordered pairs is $k(k-1)$.

$\therefore m \leq k(k-1)$.

i.e., $k^2 - k - m \geq 0$.

i.e., $k \geq \frac{1+\sqrt{4m+1}}{2}$.

$\therefore k \geq \lceil \frac{1+\sqrt{4m+1}}{2} \rceil$.

$\square$

Definition 2.2.2. *A directed path is called **unipath** if $id(v) = od(v) = 1$ for every vertex v except the first and last vertex of the directed path.*

Definition 2.2.3. *A directed cycle is called **unicycle** if $id(v) = od(v) = 1$ for any vertex v of the directed cycle.*

In general, the proper harmonious coloring problem has been viewed as an Eulerian path decomposition of graphs. Whenever such a decomposition possible, it is possible to find the proper harmonious coloring number of graphs (Hopcroft and Krishnamoorthy 1983).

Let $D(G)$ be a symmetric digraph of an undirected graph G. Then to find a proper harmonious coloring of any digraph D with n vertices, it is sufficient to find a closed trail of length n in $D(G)$ and the number of vertices of $D(G)$ gives the proper harmonious coloring number of D.

Example:

Let $D(G) = \overleftrightarrow{K}_4$, a complete symmetric digraph with 4 vertices.
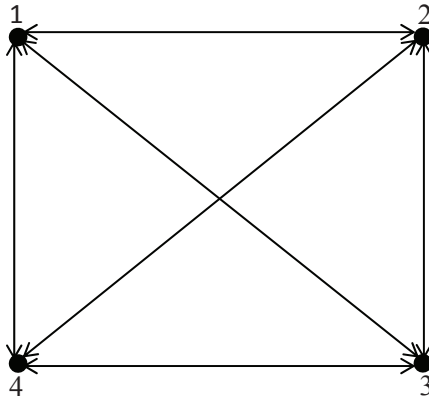
Figure 2.2: Complete symmetric digraph $\overleftrightarrow{K}_4$.

Let the closed trail be $1 - 3 - 2 - 3 - 4 - 2 - 1 - 2 - 4 - 1 - 4 - 3 - 1$. Then this closed trail can be used to find the proper harmonious coloring of the unipath $\overrightarrow{P}_{13}$ with 13 vertices and unicycle $\overrightarrow{C}_{12}$ with 12 vertices. (See Figure 2.3)

Thus, to find a proper harmonious coloring of any digraph D with n vertices, it is sufficient to find a closed trail traversing through all the arcs at least once of length n in $D(G)$ and the number of vertices of $D(G)$ gives the proper harmonious number of D.
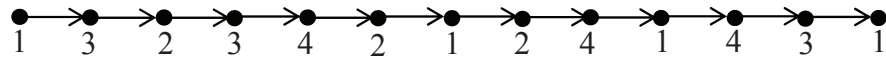
(a)
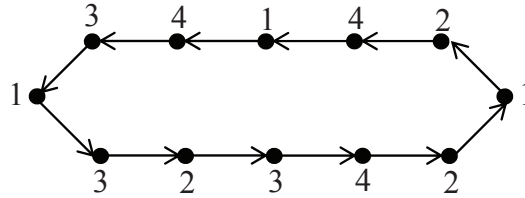
(b)

Figure 2.3: A proper harmonious coloring of (a) unipath $\overrightarrow{P_{13}}$; (b) unicycle $\overrightarrow{C_{12}}$.

In the next section, we determine the proper harmonious coloring number of some familiar classes of digraphs.

2.3 Proper harmonious coloring number of some classes of digraphs

We first consider the proper harmonious coloring number of a unipath $\overrightarrow{P}_n$ on n vertices. We start with the following result for a unipath on n vertices.

Theorem 2.3.1. *Let $\overrightarrow{P}_n$ be a unipath with n vertices. Then $\overrightarrow{\chi_h}(\overrightarrow{P}_n) = \lceil \frac{1+\sqrt{1+4(n-1)}}{2} \rceil$.*

Proof. Since unipath $\overrightarrow{P}_n$ contains $(n-1)$ arcs, $\overrightarrow{\chi_h}(\overrightarrow{P}_n) \geq \lceil \frac{1+\sqrt{1+4(n-1)}}{2} \rceil$. Let $k = \lceil \frac{1+\sqrt{1+4(n-1)}}{2} \rceil$. Then, $(k-1)(k-2) + 1 < n \leq k(k-1) + 1$.
Consider a complete symmetric digraph $\overleftrightarrow{K}_k$ with k vertices. Then $\overleftrightarrow{K}_k$ contains $k(k-1)$ arcs. To find the proper harmonious coloring of $\overrightarrow{P}_n$, it is sufficient to find an Eulerian path of length $(n-1)$ traversing through the arcs of $\overleftrightarrow{K}_k$, where $(k-1)(k-2) + 1 < n \leq k(k-1) + 1$.

We need to prove that there exists an Eulerian path of length $k(k-1)$. We shall prove this by mathematical induction. For $k = 2$ the result holds. Assume that the result is true for $k = m$. i.e. there exists an Eulerian path of length $m(m-1)$ in $\overleftrightarrow{K}_m$. Consider $\overleftrightarrow{K}_m$ and a vertex v. Then joining v to all the vertices of $\overleftrightarrow{K}_m$ in both directions, we get $\overleftrightarrow{K}_{m+1}$. Let $u_1, u_2, ..., u_m$ be the vertices of $\overleftrightarrow{K}_m$. Let u_m be the end vertex of the Eulerian path of length $m(m-1)$ (Consequently it is the first vertex). Then traverse along the path $u_m \ v \ u_1 \ v \ u_2 \ v...u_{m-1} \ v \ u_m$ and see that it is the extension of the Eulerian path obtained from $\overleftrightarrow{K}_m$ (of length $m(m-1)$), so that the length of the path obtained is $m(m-1) + 2m = m(m+1)$.

Hence by the principle of mathematical induction, the result holds.

$\square$

Figure 2.4 is an illustration of the above result.
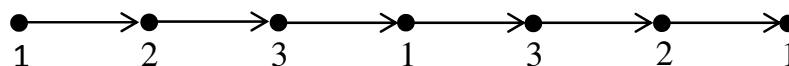
Figure 2.4: A proper harmonious coloring of $\overrightarrow{P}_7$.

Theorem 2.3.2. *Let $D = \overrightarrow{P}_1 \bigcup \overrightarrow{P}_2 \bigcup \cdots \bigcup \overrightarrow{P}_i$ be a union of disjoint unipaths, where $\overrightarrow{P}_j$ has j vertices for $j = 1, 2, \cdots, i$. Then $\overrightarrow{\chi_h}(D) = k = \lceil \frac{1+\sqrt{2i^2-2i+1}}{2} \rceil$.*

Proof. Let $D = \overrightarrow{P}_1 \bigcup \overrightarrow{P}_2 \bigcup \cdots \bigcup \overrightarrow{P}_i$. Then D has $\frac{i(i+1)}{2}$ vertices and $\frac{i(i-1)}{2}$ arcs. We know that $k \geq \lceil \frac{1+\sqrt{4m+1}}{2} \rceil$, where m is the number of arcs.

$\Longrightarrow k \geq \lceil \frac{1+\sqrt{\frac{4i(i-1)}{2}+1}}{2} \rceil$

$\Longrightarrow k \geq \lceil \frac{1+\sqrt{2i^2-2i+1}}{2} \rceil.$

We shall prove that $k = \lceil \frac{1+\sqrt{2i^2-2i+1}}{2} \rceil$.

The proper harmonious coloring number of D is equivalent to the proper harmonious coloring number of a unipath $\overrightarrow{P}_t$, where $\overrightarrow{P}_t$ is the unipath obtained by adjoining the endvertex of $\overrightarrow{P}_j$ and the starting vertex of $\overrightarrow{P}_{j+1}$ for $j = 1, 2, \cdots, i - 1$. Since $\overrightarrow{P}_t$ contains $\frac{i(i-1)}{2} + 1$ vertices, $\overrightarrow{P}_t$ can be colored with $k = \lceil \frac{1+\sqrt{2i^2-2i+1}}{2} \rceil$ colors (by

Theorem 2.3.1). Let $a_1, a_2, \cdots, a_t$ be the minimal sequence of colors assigned to the vertices of unipath $\overrightarrow{P_t}$. Note that $a_1, a_2, \cdots, a_t$ are not distinct. Now assign the colors $a_{\frac{(j-1)(j-2)}{2}+1}, a_{\frac{(j-1)(j-2)}{2}+2}, \cdots, a_{\frac{j(j-1)}{2}+1}$ to the vertices of $\overrightarrow{P_j}$, for $j = 1, 2, \cdots, i$. Note that the color of the end vertex of $\overrightarrow{P_j}$ $(1 \le j < i)$

$= a_{\frac{j(j-1)}{2}+1}$

$= a_{\frac{(j+1-1)(j+1-2)}{2}+1}$

$=$ the color of the starting vertex of $\overrightarrow{P}_{j+1}$ $(j < j + 1 \le i)$.

Hence, $\overrightarrow{\chi_h}(D) = \overrightarrow{\chi_h}(\overrightarrow{P_t}) = \lceil \frac{1+\sqrt{2i^2-2i+1}}{2} \rceil$.

$\square$

Figure 2.5 is an illustration of the above result.
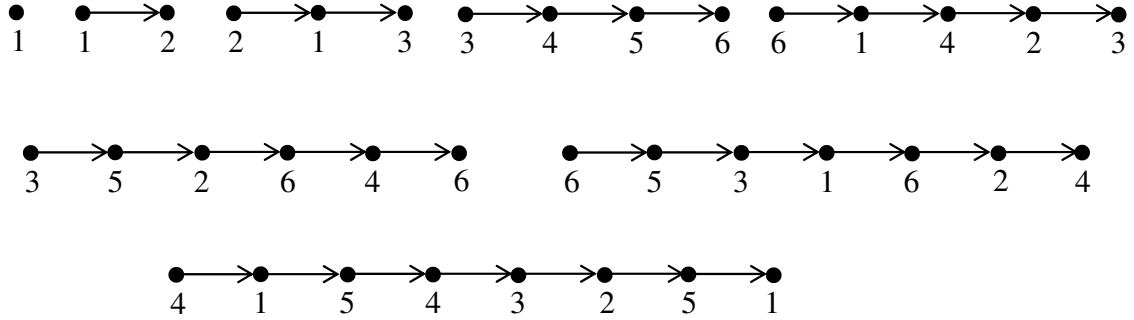
Figure 2.5: A proper harmonious coloring of union of disjoint unipaths.

Theorem 2.3.3. *Let $\overrightarrow{C}_n$ be a unicycle with n vertices, then,*

$$\overrightarrow{\chi_h}(\overrightarrow{C}_n) = \begin{cases} k + 1 & for \quad n = k(k-1) - 1, \\ k & otherwise, \end{cases}$$

where $k = \lceil \frac{1+\sqrt{4n+1}}{2} \rceil$ for $(k-1)(k-2) + 1 \le n \le k(k-1)$.

Proof. Since a unicycle $\overrightarrow{C}_n$ contains n arcs, $\overrightarrow{\chi_h}(\overrightarrow{C}_n) = k \ge \lceil \frac{1+\sqrt{4n+1}}{2} \rceil$. Let $k = \lceil \frac{1+\sqrt{4n+1}}{2} \rceil$. Then $(k-1)(k-2) + 1 \le n \le k(k-1)$. It is equivalent to prove that there exists an Eulerian circuit with n arcs and k vertices for $(k-1)(k-2) + 1 \le n \le k(k-1)$ except for $n = k(k-1) - 1$. We know that a digraph D has an Eulerian circuit if and only if $id\,(v) = od\,(v)$ for

every vertex v. For $n = k(k-1)-1$, the possible degree sequence (k vertices) is $(k-1), (k-1), ..., (k-1), (k-2)$. But there exists no such digraph. (For otherwise, let v be the vertex with $id\,(v) = od\,(v) = k-2$. Since degree of each of the other vertex is $k-1$, every other vertex has an arc to v so that $id\,(v) = k-1$, a contradiction.) Thus, we require one more additional color to color the vertices of $\overrightarrow{C}_n$ when $n = k(k-1)-1$. Hence $\overrightarrow{\chi_h}(\overrightarrow{C}_n) = k+1$, for $n = k(k-1)-1$.

Consider a complete symmetric digraph $\overleftrightarrow{K}_k$ with k vertices. In $\overleftrightarrow{K}_k$, $id\,(v) = od\,(v) = k-1$ for all v. Hence $\overleftrightarrow{K}_k$ is Eulerian. Therefore, we have the result for $n = k(k-1)$. Remove an Eulerian cycle of length i, where $i = 2, 3, 4, ..., (2k-3)$ from $\overleftrightarrow{K}_k$. Then we get an Eulerian cycle of length $k(k-1)-2, k(k-1)-3, ..., (k-1)(k-2)+1$. Since we are removing a cycle, the equation $id\,(v) = od\,(v)$ remains unchanged for the vertices lying on the cycle. (When we remove an outgoing arc, we remove an incoming arc and vice versa.)

Therefore, the resulting cycle is also an Eulerian circuit. Hence $\overrightarrow{\chi_h}(\overrightarrow{C}_n) = k$, for $(k-1)(k-2)+1 \le n \le k(k-1)$ except for $n = k(k-1)-1$.

$\square$

Figures 2.6 and 2.7 are the illustrative examples of the above result.
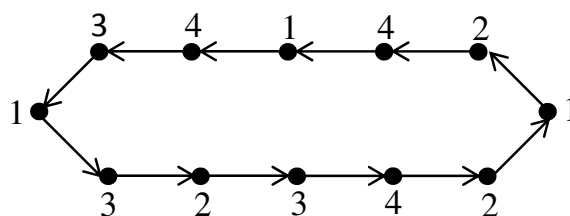
Figure 2.6: A proper harmonious coloring of $\overrightarrow{C}_{12}$.

Theorem 2.3.4. *Let $D = \overrightarrow{C}_3 \bigcup \overrightarrow{C}_4 \bigcup \cdots \bigcup \overrightarrow{C}_i$ be a union of disjoint unicycles, where $\overrightarrow{C}_j$ has j vertices for $j = 3, 4, \cdots, i$. Then $\overrightarrow{\chi_h}(D) = k = \lceil \frac{1+\sqrt{2i^2+2i-11}}{2} \rceil$.*

Proof. Let $D = \overrightarrow{C}_3 \bigcup \overrightarrow{C}_4 \bigcup \cdots \bigcup \overrightarrow{C}_i$. Then D has $\frac{(i-2)(i+3)}{2}$ vertices and $\frac{(i-2)(i+3)}{2}$ arcs.
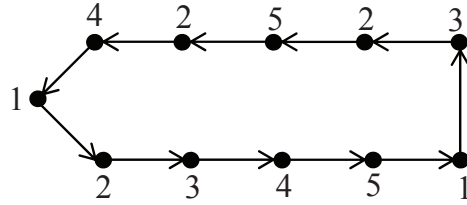
Figure 2.7: A proper harmonious coloring of $\overrightarrow{C_{11}}$.

We know that $k \geq \lceil \frac{1+\sqrt{4m+1}}{2} \rceil$, where m is the number of arcs.

$\implies k \geq \lceil \frac{1+\sqrt{\frac{4(i-2)(i+3)}{2}+1}}{2} \rceil$

$\implies k \geq \lceil \frac{1+\sqrt{2i^2+2i-11}}{2} \rceil = t.$

To prove that $k = t$.

Consider the complete symmetric digraph $\overleftrightarrow{K}_t$. Since $\overleftrightarrow{K}_t$ is Eulerian, it can be partitioned into cycles (from **Theorem 4.4** of Chartrand and L.Lesniak (2004)). It can be proved by induction that $\overleftrightarrow{K}_t$ can be partitioned such that the partition include cycles of length $3,4,...,i$. The vertices of these cycles give the harmonious coloring of D. Hence the harmonious coloring number of D is $k \leq t$.

$\therefore$ $k = t$.

i.e. $\overrightarrow{\chi_h}(D) = \lceil \frac{1+\sqrt{2i^2+2i-11}}{2} \rceil$. $\qquad \square$

Figure 2.8 is an illustration of the above result.
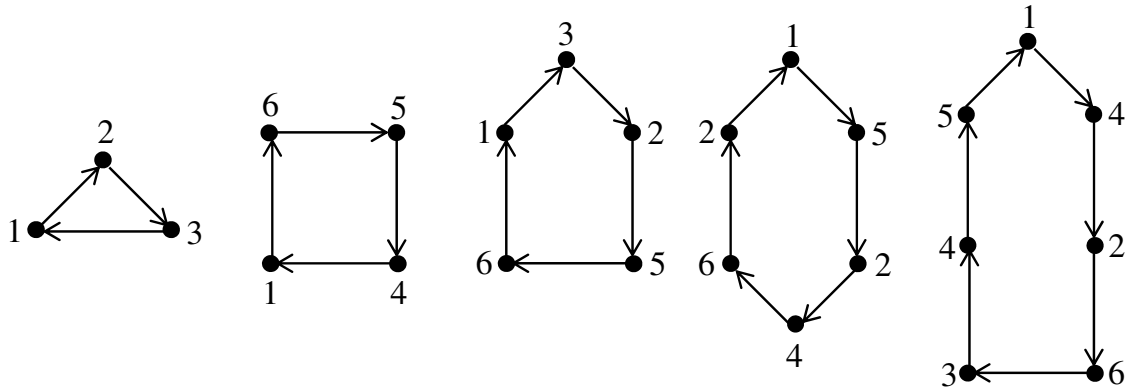
Figure 2.8: A proper harmonious coloring of union of disjoint unicycles.

Theorem 2.3.5. *Let $\overleftrightarrow{C}_n$ be a symmetric cycle with n vertices. Then, $\overrightarrow{\chi_h}(\overleftrightarrow{C}_n) \geq$*

$k = \lceil \frac{1+\sqrt{8n+1}}{2} \rceil$. *In particular,*

$$\overrightarrow{\chi_h}(\overleftrightarrow{C}_n) = \begin{cases} n & for \quad n = 3, 4. \\[2mm] k & for \quad k^2 - 4k + 5 \leq 2n \leq k(k-1) \quad k \geq 5 \text{ and } k \text{ is odd.} \\[2mm] k+1 & for \quad 2n = k(k-1) - j, j = 2, 4 \quad k \geq 5 \text{ and } k \text{ is odd.} \\[2mm] k & for \quad k^2 - 3k + 4 \leq 2n \leq k(k-2) \quad k \geq 6 \text{ and } k \text{ is even.} \end{cases}$$

It is similar to the harmonious coloring number of undirected cycle which is proved by Frank et al. (1982).

Figure 2.9, Figure 2.10, Figure 2.11 and Figure 2.12 are the illustrative examples of the above result.
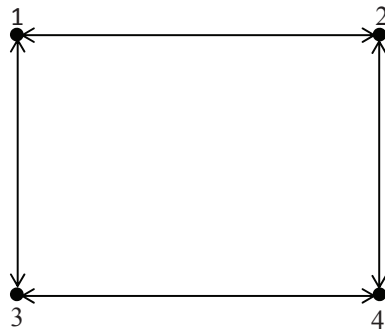
Figure 2.9: A proper harmonious coloring of $\overleftrightarrow{C}_4$.

Figure 2.10: A proper harmonious coloring of $\overleftrightarrow{C}_9$.

Figure 2.11: A proper harmonious coloring of $\overleftrightarrow{C_{10}}$.

Figure 2.12: A proper harmonious coloring of $\overleftrightarrow{C_{12}}$.

Definition 2.3.6. *A directed graph in which the underlying graph is a star is known as a* **directed star***.*

Theorem 2.3.7. $\overrightarrow{\chi_h}(\overrightarrow{S}_n) = max[id\ (v), od\ (v)] + 1$*, where* $\overrightarrow{S}_n$ *is a directed star with n vertices and v is the central vertex.*

Proof. Let $\overrightarrow{S}_n$ be a directed star where n is the number of vertices. Let v be the central vertex. Let there be s incoming arcs to v and t outgoing arcs from v. Label the central vertex v as 1.

Case (i) Let $s > t$. Then the incoming arcs to v will be $(2, 1), (3, 1), ..., (s+1, 1)$ and the outgoing arcs from v will be $(1, 2), (1, 3), ..., (1, t)$.

$\therefore \overrightarrow{\chi_h}(\overrightarrow{S}_n) = s + 1$.

Case (ii) Let $t > s$. Then the outgoing arcs from v will be $(1, 2), (1, 3), ..., (1, t+1)$ and the incoming arcs to v will be $(2, 1), (3, 1), ..., (s, 1)$.

$\therefore \overrightarrow{\chi_h}(\overrightarrow{S}_n) = t + 1$.

From the above two cases, we can conclude that

$\overrightarrow{\chi_h}(\overrightarrow{S}_n) = max[id\ (v), od\ (v)] + 1$. $\qquad\qquad\square$

Figure 2.13 is an illustration of the above result.
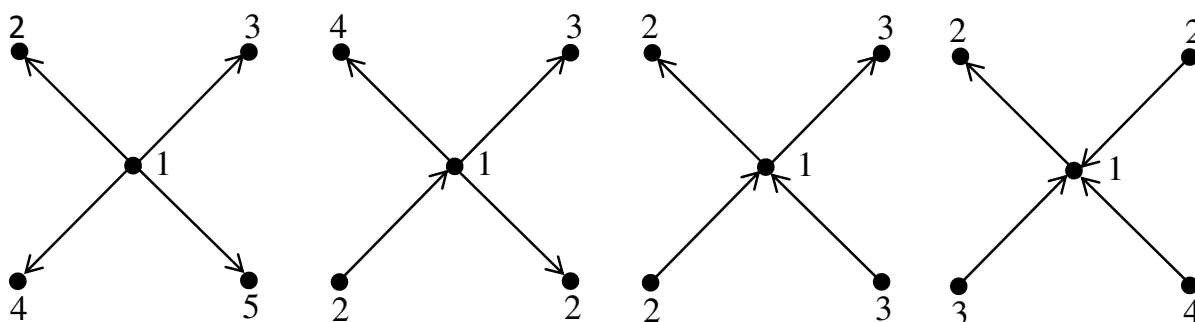
Figure 2.13: Proper harmonious coloring of $\overrightarrow{S_5}$.

Definition 2.3.8. *A* **directed wheel** $\overrightarrow{W_n}$ *with n vertices is a graph obtained from the directed cycle* $\overrightarrow{C}_{n-1}$ *and* K_1, *by joining every vertex of* $\overrightarrow{C}_{n-1}$ *to the vertex of* K_1, *where the directed cycle* $\overrightarrow{C}_{n-1}$ *is called the* **rim** *and the arcs joining to* K_1 *are called the* **spokes**. *The vertex of* K_1 *is called the* **central vertex**. *If the rim of a directed wheel is a unicycle, then the wheel is called* **unicyclic**.

Theorem 2.3.9. *Let* $\overrightarrow{W_n}$ *be a unicyclic wheel with n vertices and let v be the central vertex. Then*

(i) $\overrightarrow{\chi_h}(\overrightarrow{W}_4) = (id\ (v) + od\ (v) + 1)$.

(ii) For n = 5 and 6

$$
\overrightarrow{\chi_h}(\overrightarrow{W}_n) = \begin{cases} n & if\ id\ (v) = 0\ or\ od\ (v) = 0, \\ id\ (v) + od\ (v) & otherwise. \end{cases}
$$

(iii) For $n \geq 7$, $\overrightarrow{\chi_h}(\overrightarrow{W}_n) = max[id\ (v), od\ (v)] + 1$.

Proof. (i) and (ii) can be easily verified.

(iii) Let $\overrightarrow{W_n}$ be a unicyclic wheel, where $n \geq 7$. The total number of arcs of the wheel is $2(n-1)$. Let $v = v_1$ be the central vertex and let $v_2, v_3, ..., v_n$ be the vertices on the circumference of the wheel. Let there be s incoming arcs to v and t outgoing arcs from v. Label the vertex v_1 as 1.

Case (i) Let $s > t$. Label the tails of the incoming arcs to v as $2, 3, ..., s+1$ so that the incoming arcs to v will be $(2,1), (3,1), ..., (s+1,1)$ and also label the heads of the outgoing arcs from v as $2, 3, ..., s+1, \quad (s+1 > t)$ provided the adjacent vertices on the circumference of the wheel will not get the same color. Hence the outgoing arcs from v will be $(1,2), (1,3), ..., (1, s+1)$.

$\therefore \overrightarrow{\chi_h}(\overrightarrow{W_n}) = s + 1$.

Case (ii) Let $t > s$. Label the heads of the outgoing arcs from v as $2, 3, ..., t+1$ so that the outgoing arcs from v will be $(1,2), (1,3), ..., (1, t+1)$ and also label the tails of the incoming arcs to v as $2, 3, ..., t+1, \quad (t+1 > s)$ provided the adjacent vertices on the circumference of the wheel will not get the same color. Hence the incoming arcs to v will be $(2,1), (3,1), ..., (t+1, 1)$.

$\therefore \overrightarrow{\chi_h}(\overrightarrow{W_n}) = t + 1$.

From case (i) and case (ii), it follows that

$$\overrightarrow{\chi_h}(\overrightarrow{W_n}) = max[id\ (v), od\ (v)] + 1.$$

$\square$

Figure 2.14, Figure 2.15 and Figure 2.16 are the illustrative examples of the above result.
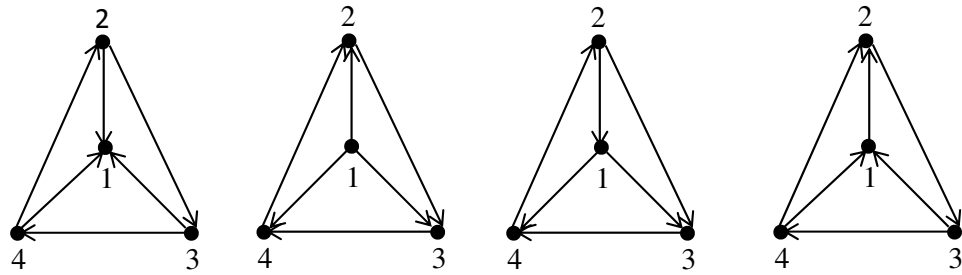
Figure 2.14: Proper harmonious coloring of $\overrightarrow{W_4}$.

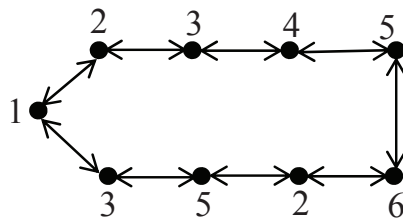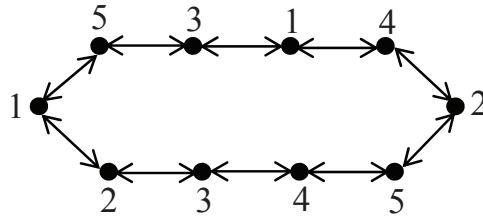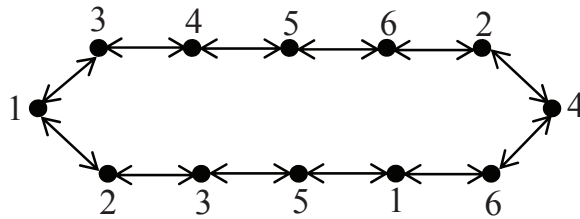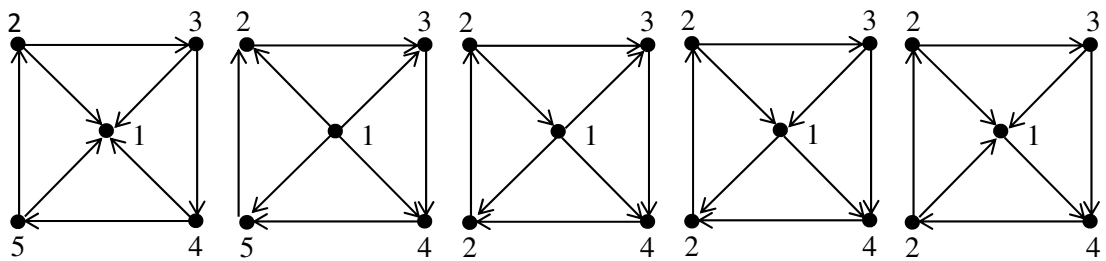Figure 2.15: Proper harmonious coloring of $\overrightarrow{W_5}$.
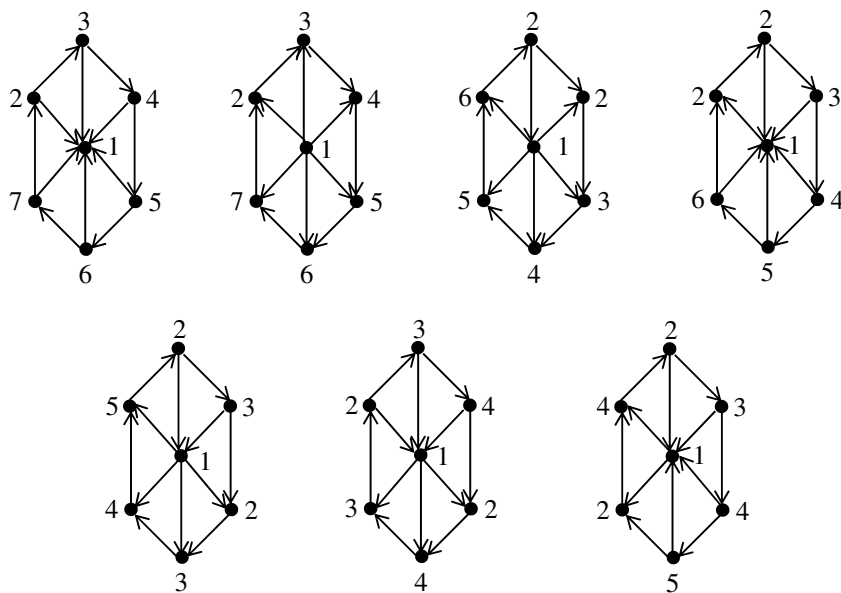
Figure 2.16: Proper harmonious coloring of $\overrightarrow{W_7}$.

Theorem 2.3.10. *For any n-ary out-tree $\overrightarrow{T}_n$,*

$$\overrightarrow{\chi_h}(\overrightarrow{T}_n) = k \leq \frac{n^{\lfloor \frac{l}{2} \rfloor + 1} - 1}{n-1}, \text{ where } l \text{ is the level of the tree.}$$

Proof. Let $\overrightarrow{T}_n$ be the n-ary out-tree of level $l, l = 1, 2, \ldots$.

It is enough to prove the result for complete n-ary out-tree $\overrightarrow{T}_n$. i.e. To prove that $\overrightarrow{\chi_h}(\overrightarrow{T}_n) = k = \frac{n^{\lfloor \frac{l}{2} \rfloor + 1} - 1}{n-1}$ for complete n-ary out-tree, $\overrightarrow{T}_n$. There are $\frac{n^l - 1}{n-1}$ vertices and $\frac{n(n^{l-1}-1)}{n-1}$ arcs in $\overrightarrow{T}_n$. We color the vertices of n-ary tree as follows:

Color the root vertex as 1. In level 2 there are n vertices. Color the vertices as $2, 3, \ldots, n+1$. Hence the total number of colors used in level 2 is $n+1$. In level 3, color the vertices as follows:

$$L(v_i) = \begin{cases} j & if \ j \leq k+1 \\ j+1 & if \ j > k+1, \end{cases}$$

where $i = kn + j, \quad k = 0, 1, 2, \ldots, n-1, \quad j = 1, 2, \ldots, n$ and v_i are the vertices of level 3. Hence in level 3, the total number of colors required is $n+1$. Now in level 4, color the vertices adjacent to 1 as $n+2, n+3, \ldots, 2n+1$; $2n+2, 2n+3, \ldots, 3n+1$; $\ldots$; $n^2 + 2, n^2 + 3, \ldots, n(n+1)+1$. Use the same colors for the vertices adjacent to $2, 3, \ldots, n+1$. Hence in level 4, n^2 additional colors are required to color the vertices. In the next level, one can observe that all the vertices adjacent to the vertices colored with $n+2, n+3, \ldots, n(n+1)+1$ can be colored as $1, 2, \ldots, n$. In this level, we don't require any additional colors to color the vertices. Continuing in this way, one can observe that the number of colors used in any odd level l is less than or equal to the number of colors used till the $l-1$ level and the graph is harmonious.

We shall prove the theorem by mathematical induction on the level l. For $l = 1$, one can easily see that $k = 1$.

Assume that the result is true for some level $l = m$.

i.e., $k = \frac{n^{\lfloor \frac{m}{2} \rfloor + 1} - 1}{n-1}$.

Now, to prove that the result is true for $l = m + 1$,

i.e., to prove that $k = \frac{n^{\lfloor \frac{m+1}{2} \rfloor + 1} - 1}{n-1}$,

we shall consider 2 cases.

Case (i) Let $l = m$ be even. Then $m + 1$ is odd. Also, the number of colors used in any odd level is less than or equal to the number of colors used till the previous level (i.e. even level) except for level 1. Also, the number of colors added in level m is equal to $n^{\frac{m}{2}}$. We know that the number of colors sufficient in level $m + 1$ is equal to the number of colors used in level m as m is even.

Now, by induction hypothesis, the number of colors used in level m is $k = \frac{n^{\lfloor \frac{m}{2} \rfloor + 1} - 1}{n - 1}$. Here, $\lfloor \frac{m+1}{2} \rfloor = \lfloor \frac{m}{2} \rfloor$ as m is even. Hence the number of colors used in level $m + 1$ will be, $k = \frac{n^{\lfloor \frac{m+1}{2} \rfloor + 1} - 1}{n - 1}$.

Case (ii) Let $l = m$ be odd. Then $m + 1$ is even. The number of colors added in level m is $n^{\frac{m-1}{2}}$. Hence the number of colors added in level $m + 1$ is $n^{\frac{m+1}{2}}$. Also, by induction hypothesis, the number of colors used in level m is $k = \frac{n^{\lfloor \frac{m}{2} \rfloor + 1} - 1}{n - 1}$.

Hence the number of colors used in level $m + 1$ is

$k = \frac{n^{\lfloor \frac{m}{2} \rfloor + 1} - 1}{n - 1} + n^{\frac{m+1}{2}}$

$\Rightarrow k = \frac{n^{(\frac{m-1}{2} + 1)} - 1}{n - 1} + n^{\frac{m+1}{2}}$ (as m is odd, $\lfloor \frac{m}{2} \rfloor = \frac{m-1}{2}$)

i.e., $k = \frac{n^{\frac{m+3}{2}} - 1}{n - 1}$

i.e., $k = \frac{n^{\frac{m+1}{2} + 1} - 1}{n - 1}$

i.e., $k = \frac{n^{\lfloor \frac{m+1}{2} \rfloor + 1} - 1}{n - 1}$ [as m is odd, $\frac{m+1}{2} = \lfloor \frac{m+1}{2} \rfloor$].

Therefore, the result is true for level $l = m + 1$.

Hence, by the principle of mathematical induction, the result follows for complete n-ary out-tree $\overrightarrow{T}_n$ of level l. This proves that for any n-ary out-tree $\overrightarrow{T}_n$

$\overrightarrow{\chi_h}(\overrightarrow{T}_n) = k \leq \frac{n^{\lfloor \frac{l}{2} \rfloor + 1} - 1}{n - 1}$.

$\square$

Figure 2.17 is an illustration of the above result.

Definition 2.3.11. *An **alternating Eulerian trail** of a digraph D is an open trail of D including all the arcs and vertices of D such that any two arcs consecutive on the trail have opposite direction.*
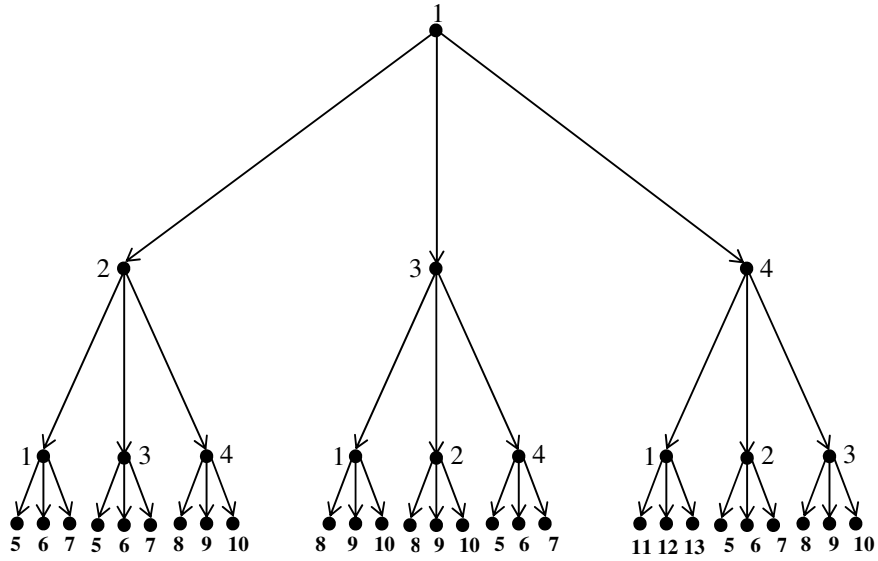
Figure 2.17: A proper harmonious coloring of $\overrightarrow{T_3}$ of level 4.

Lemma 2.3.12. *If G is a connected (undirected) non-bipartite graph in which every vertex has even degree, then the symmetric digraph $D(G)$ obtained from G has an alternating closed Eulerian trail.*

Proof. Let G be a connected (undirected) non-bipartite graph in which every vertex has even degree. Since G is non-bipartite, it has a cycle of odd length. Also, since all the vertices of G are of even degree, G is Eulerian. That is G has a closed Eulerian trail say, T. Consider the symmetric digraph $D(G)$ obtained from G by replacing each undirected edge by a pair of edges with opposite orientations. Since G has a closed Eulerian trail T, we get a closed Eulerian trail in $D(G)$ in which the adjacent edges have opposite direction as follows:

Suppose G has m edges. Then there exists two cases.

Case (i) Let m be odd.

We obtain the required alternating closed Eulerian trail in $D(G)$ by traversing T twice in the same direction but using the edges of alternating direction.

Case (ii) Let m be even.

We observe that T must visit some vertex v twice with an odd number of edges between the visits (otherwise G would be bipartite). Hence we get T as $v_0 = v, v_1, \cdots, v_i = v, v_{i+1}, \cdots, v_m = v$, where i is odd. Then the required alternating closed Eulerian trail is $v_0 = v \to v_1 \leftarrow \cdots \to v_i = v \leftarrow v_1 \to \cdots \leftarrow v_i = v \to v_{i+1} \leftarrow \cdots \to v_m = v \leftarrow v_{i+1} \to \cdots \leftarrow v_m = v$.

$\square$

As a consequence of the above lemma, we have the following lemma.

Lemma 2.3.13. *The alternating cycle on n vertices can be colored with k colors if there is a connected (undirected) non-bipartite graph G, with every vertex having even degree and with k vertices and $\frac{n}{2}$ edges.*

ALTERNATING PATHS

Definition 2.3.14. *An **alternating path** $\overrightarrow{AP_n}$ with n vertices is an oriented path in which any two arcs consecutive on the path have opposite direction.*

Theorem 2.3.15. *Let $\overrightarrow{AP_n}$ be an alternating path. Then*

$$\overrightarrow{\chi_h}(\overrightarrow{AP_n}) = \begin{cases} k+1 & \text{if } k \text{ is even and } k^2 - 2k + 3 \le n \le k^2 - k + 1 \\ k & \text{otherwise,} \end{cases}$$

where $k = \lceil \frac{1 + \sqrt{1 + 4(n-1)}}{2} \rceil$.

Proof. Since $\overrightarrow{AP_n}$ is an alternating path with n vertices and $n-1$ arcs, by Theorem 2.2.1, we get $\overrightarrow{\chi_h}(\overrightarrow{AP_n}) \ge k$. When $\overrightarrow{\chi_h}(\overrightarrow{AP_n}) = k$, it follows that $k^2 - 3k + 4 \le n \le k^2 - k + 1$.

Let G be a connected (undirected) non-bipartite graph in which every vertex has even degree. Then to find the proper harmonious coloring of $\overrightarrow{AP_n}$, it is sufficient to find an alternating Eulerian trail of length n, by traversing in the same direction but using

arcs of opposite direction in $D(G)$. Consider a complete undirected graph K_k with k vertices.

Case (i) Let k be odd and let $G = K_k$. Then G contains $\frac{k(k-1)}{2}$ edges. Since G is non-bipartite and all the vertices are of even degree, G has an undirected closed Eulerian trail T. Then by Lemma 2.3.12, we obtain the required alternating closed Eulerian trail in $D(G)$.

Case (ii) Let k be even.

Case (a) Let $k = 4$ and let v_1, v_2, v_3 and v_4 be the vertices of K_4. Then we can find an alternating Eulerian trail in $\overleftrightarrow{K_4}$ as follows: $v_1 \rightarrow v_2 \leftarrow v_3 \rightarrow v_1 \leftarrow v_2 \rightarrow v_4 \leftarrow v_1 \rightarrow v_3 \leftarrow v_4 \rightarrow v_1$.

Case (b) Let $k \geq 6$ and let $G = K_k \setminus M$, where M is the matching of size $k/2$ (k should be at least 6 so that G is not bipartite). Then G will have k vertices and $\frac{k^2-2k}{2}$ edges. Also, all the vertices of G are of even degree. Hence G has an undirected closed Eulerian trail T. As m will be even for any value of k, by Lemma 2.3.12, we can find an alternating closed Eulerian trail in $D(G)$ and the length of this alternating closed Eulerian trail is $2(\binom{k}{2} - k/2) = k^2 - 2k$. Regarding this as an open alternating trail, we can clearly add one further arc to one end of it in $D(G)$ using one edge of the matching in one direction. Hence we will get an alternating Eulerian trail of length $k^2 - 2k + 1$.

We know that since k colors are used to color the vertices of $\overrightarrow{AP_n}$ of length $n - 1$, there will be $k(k - 1)$ ordered pairs of colors. In $\overrightarrow{AP_n}$, at each vertex there will be either two incoming arcs or two outgoing arcs except for the first and the last vertex. Hence it requires even number of ordered pairs at each vertex. There will be $k - 1$ ordered pairs associated with each color. When k is even, $k-1$ will be odd and hence we cannot use $k - 1$ ordered pairs of one particular color. That is only $k^2 - 2k + 1$ ordered pairs of colors will be used when k is even. Hence when k is even, for an alternating path with more than $k^2 - 2k + 2$ vertices, we require $k + 1$ colors. Hence the proof.

$\square$

Figures 2.18, 2.19 and 2.20 are the illustrative examples of the above result.
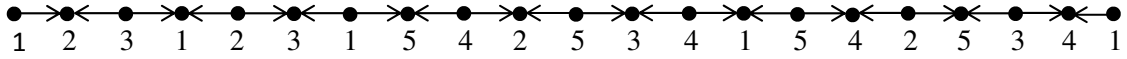
Figure 2.18: A proper harmonious coloring of $\overrightarrow{AP}_{21}$.
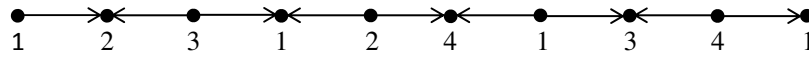
Figure 2.19: A proper harmonious coloring of $\overrightarrow{AP}_{10}$.
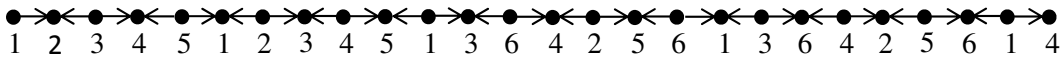
Figure 2.20: A proper harmonious coloring of $\overrightarrow{AP}_{26}$.

ALTERNATING CYCLES

Definition 2.3.16. *An **alternating cycle** $\overrightarrow{AC}_n$ with n vertices is an oriented cycle in which any two arcs consecutive on the cycle have opposite direction.*

Let $k = \lceil \frac{1+\sqrt{4n+1}}{2} \rceil$, where n is the number of vertices of the alternating cycle $\overrightarrow{AC}_n$. Let $\overleftrightarrow{K}_k$ be a complete symmetric digraph. Then we have the following results:

Lemma 2.3.17. *Let the alternating cycle $\overrightarrow{AC}_n$ be a subgraph of $\overleftrightarrow{K}_k$ of length n. Then every vertex of $\overrightarrow{AC}_n$ in $\overleftrightarrow{K}_k$ has even indegree and outdegree.*

Proof. Let $\overrightarrow{AC}_n$ be any alternating cycle of length n in $\overleftrightarrow{K}_k$. By definition, an alternating cycle is a cycle in which any two consecutive arcs have opposite directions. Hence any vertex of $\overrightarrow{AC}_n$ in $\overleftrightarrow{K}_k$ should have either two incoming arcs or two outgoing arcs. Hence every vertex of $\overrightarrow{AC}_n$ will have even indegree and outdegree. Hence the proof.

$\square$

Lemma 2.3.18. *Let the alternating cycle $\overrightarrow{AC_n}$ be a subgraph of $\overleftrightarrow{K_k}$. When k is odd, for $n = k(k-1) - 2$, $\overrightarrow{AC_n}$ cannot be colored with k colors and hence requires $k + 1$ colors.*

Proof. Let us assume that $k(k-1)-2$ vertices can be colored with k colors. Then in $\overleftrightarrow{K_k}$, the possible degree sequence of the outgoing arcs will be $(k-1, k-1, ...(k-1)times, k-3)$. Then corresponding to this degree sequence of outgoing arcs, we get the degree sequence of the incoming arcs as $(k-1, k-1, ...(k-2)times, k-2, k-2)$. Hence there exists at least two vertices having odd indegrees, a contradiction by Lemma 2.3.17. Thus, at least $k + 1$ colors are required.

$\square$

Lemma 2.3.19. *Let $\overrightarrow{AC_n}$ be an alternating cycle with n vertices. Then when k is even, $k(k-2) + 2 \leq n \leq k(k-1)$ vertices cannot be colored with k colors and hence requires $k + 1$ colors.*

Proof. The total indegree and the total outdegree of those vertices of the alternating cycle having any particular color must both be even (by Lemma 2.3.17) and so if k is even, they cannot exceed $(k-2)$ (as there are only $(k-1)$ ordered pairs of one particular color). It follows that there can be at most $k(k-2)$ arcs in the alternating cycle with k colors. Hence for $\overrightarrow{AC_n}$ with n vertices, where $k(k-2)+2 \leq n \leq k(k-1)$, we require one more additional color to color the vertices.

$\square$

Theorem 2.3.20. *Let* $\overrightarrow{AC_n}$ *be an alternating cycle with n vertices, where n is even.*

Then

$$\overrightarrow{\chi_h}(\overrightarrow{AC_n}) = \begin{cases} k & for\ k = odd\ and\ n = (k-1)(k-2)+2, ..., k(k-1)-4, k(k-1) \\ k+1 & for\ k = odd\ and\ n = k(k-1)-2 \\ k & for\ k = even\ and\ n = (k-1)(k-2)+2, ..., k(k-2) \\ k+1 & for\ k = even\ and\ n = k(k-2)+2, k(k-2)+4, ..., k(k-1). \end{cases}$$

Proof. Since an alternating cycle $\overrightarrow{AC_n}$ contains n arcs, by Theorem 2.2.1, we get $\overrightarrow{\chi_h}(\overrightarrow{AC_n}) \geq k$. When $\overrightarrow{\chi_h}(\overrightarrow{AC_n}) = k$, it follows that $(k-1)(k-2) + 2 \leq n \leq k(k-1)$. Let G be a connected (undirected) non-bipartite graph in which every vertex has even degree. Consider a complete undirected graph K_k with k vertices.

Case (i) Let k be odd.

Case (a) Let $n = k(k-1)$, $k \geq 3$ and let $G = K_k$. Then G contains $\frac{k(k-1)}{2}$ edges. Since G is non-bipartite and all the vertices are of even degree, G has an undirected closed Eulerian trail T. Then by Lemma 2.3.13, we can find an alternating cycle of length n.

Case (b) Let $n = (k-1)(k-2) + 2, \cdots, k(k-1) - 6,\ \ k \geq 5$ and let $G = K_k \setminus C_t$, where C_t is a cycle with t vertices, $t = 3, 4, \cdots, k-2$. Then G has $\frac{k(k-1)-2t}{2}$ edges, where $t = 3, 4, \cdots, k-2$. Also, G is non-bipartite and all the vertices of G are of even degree. Hence G has an undirected closed Eulerian trail T. Then by Lemma 2.3.13, we obtain the required alternating cycle of length n.

Case (c) Let $n = k(k-1) - 4,\ \ k \geq 5$. Let $v_1, v_2, v_3, \cdots, v_k$ be the vertices of K_k. Let $G = K_k \setminus C_4$, where C_4 is a cycle v_1, v_2, v_3, v_4, v_1 of length 4. Then G has $\frac{k(k-1)}{2} - 4$ edges. Also, G is non-bipartite and all the vertices of G are of even degree. Hence G has an undirected closed Eulerian trail T. Then by Lemma 2.3.13, we obtain the alternating closed Eulerian trail of length $k(k-1) - 8$ in $D(G)$. Suppose the alternating closed Eulerian trail contains $\cdots \to v_1 \leftarrow \cdots$, and add in the arcs $v_1 \leftarrow v_2 \to v_3 \leftarrow v_4 \to v_1$. Then we obtain the required alternating closed Eulerian

trail of length $k(k-1) - 4$.

Case (ii) Let k be even and let $G_1 = K_k \setminus M_1$, where M_1 is the matching of size $k/2$ (k should be at least 6 so that G_1 is not bipartite).

Case (a) When $n = k(k-2)$ and $k = 4$, we can color the vertices of $\overrightarrow{AC_8}$ as given below:
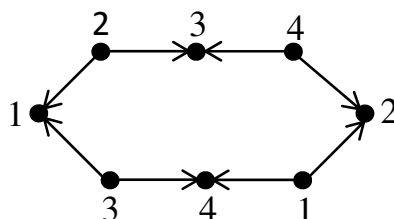
Figure 2.21: A proper harmonious coloring of $\overrightarrow{AC_8}$.

Case (b) Let $n = k(k-2)$, $k \geq 6$ and let $G = G_1$. Then G is non-bipartite and all the vertices of G are of even degree. Hence G has an undirected closed Eulerian trail T. Also G contains $\frac{k^2 - 2k}{2}$ edges. Then by Lemma 2.3.13, we obtain the required alternating cycle of length $k^2 - 2k$.

Case (c) Let $n = (k-1)(k-2) + 2, (k-1)(k-2) + 4, \cdots, k(k-1) - 6, \quad k \geq 10$ and let $G = G_1 \setminus C_t$, where C_t is a cycle with t vertices, $t = 3, 4, \cdots, (\frac{k}{2} - 2)$. Then G has $\frac{k^2 - 2(k-t)}{2}$ edges for $t = 3, 4, \cdots, (\frac{k}{2} - 2)$. Also, G is non-bipartite and all the vertices of G are of even degree. Hence G has an undirected closed Eulerian trail T. Then by Lemma 2.3.13, we obtain the required alternating cycle of length n.

Case (d) Let $n = k(k-2) - 4, \quad k \geq 8$ and let $G_2 = K_k \setminus M_2$, where M_2 is the matching of size $\frac{k}{2} - 2$ (k should be at least 6 so that G_2 is not bipartite). Then $G = G_2 \setminus 2P_3$, where P_3 is a path of length 2 and the end vertices of both the paths are the vertices which are not the adjacent vertices of the edges of the matching. Also, both the paths are distinct and passes through the vertex which is incident with the edge of the matching. The following sketch illustrates G when $k = 8$.
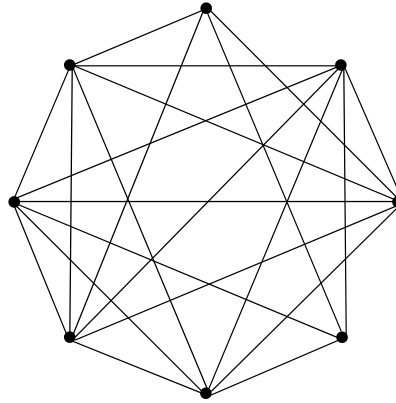
Then G has $\frac{k(k-1)-k-4}{2}$ edges. Also, G is non-bipartite and all the vertices of G are of even degree. Hence G has an undirected closed Eulerian trail T. Then by Lemma 2.3.13, we obtain the required alternating cycle of length n.

Case(e) Let $n = k(k-2)-2, \;\; k \geq 6$ and let $G_3 = K_k \setminus M_3$, where M_3 is the matching of size $\frac{k}{2} - 1$ (k should be at least 6 so that G_3 is not bipartite). Then $G = G_3 \setminus P_3$, where P_3 is a path of length 2 and the end vertices of P_3 are the vertices which are not the adjacent vertices of the edges of the matching. Also, it passes through the vertex which is incident with the edge of the matching. Consider the sketch below as an example of G for the case when $k = 6$.
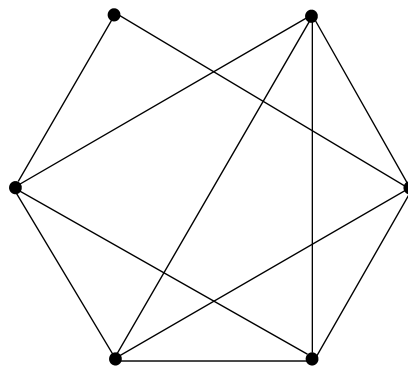
Then G has $\frac{k(k-1)-k-2}{2}$ edges. Also, G is non-bipartite and all the vertices of G are of even degree. Hence G has an undirected closed Eulerian trail T. Then by Lemma 2.3.13, we obtain the required alternating alternating cycle of length n.

We can conclude the result using Lemma 2.3.18 and Lemma 2.3.19.

□

Figures 2.22, 2.23, 2.24 and 2.25 are the illustrative examples of the above result.
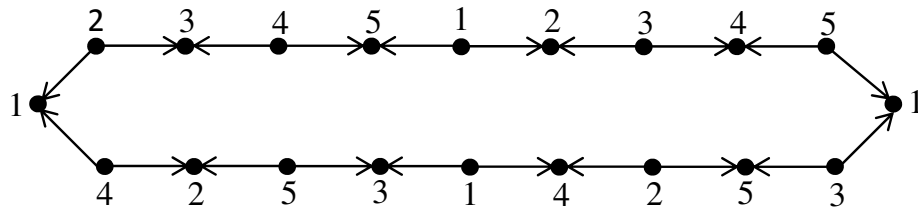
Figure 2.22:   A proper harmonious coloring of $\overrightarrow{AC}_{20}$.
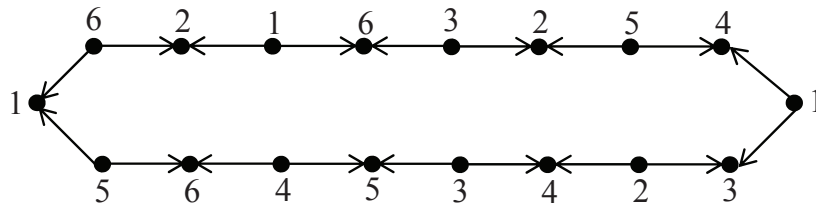
Figure 2.23:   A proper harmonious coloring of $\overrightarrow{AC}_{18}$.
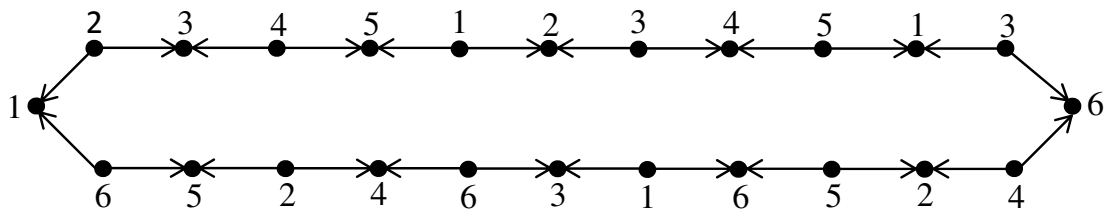
Figure 2.24:   A proper harmonious coloring of $\overrightarrow{AC}_{24}$.
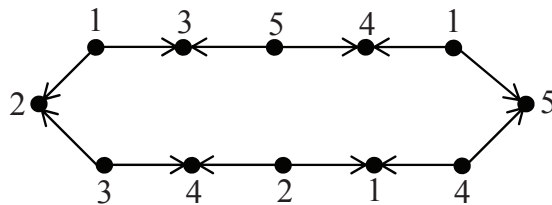
Figure 2.25:   A proper harmonious coloring of $\overrightarrow{AC}_{12}$.

Chapter 3

Harmonious Colorings of Regular Digraphs

Though determining the proper harmonious coloring number is NP-hard in general, there are families of digraphs for which this number is easy to calculate and the corresponding optimal coloring is well known.

In this chapter, we obtain a lower bound for the proper harmonious coloring number of regular digraphs and investigate the same for some classes of regular digraphs.

3.1   Introduction

In this chapter, we have further extended the concept of proper harmonious coloring number of a directed graph to all oriented graphs of an underlying undirected graph as follows:

Let G be an underlying undirected graph. Let O be an orientation of the edges of G. Denote the directed graph with orientation O as $G(O)$. Since there are $2^{|E|}$ edges, there exist $2^{|E|}$ different directed graphs say $G(O_1), G(O_2), G(O_3), ..., G(O_{2^{|E|}})$. Let h_i denote the harmonious coloring number of $G(O_i)$, $1 \leq i \leq 2^{|E|}$. Define $h = min_i \, h_i$ as the harmonious coloring number of the oriented graph $O(G)$ and denote it by $\overrightarrow{\chi_h}(O(G))$.

The following theorem gives a lower bound for $\overrightarrow{\chi_h}(D)$ which we have already proved in Chapter 2.

Theorem 3.1.1. *For any digraph D, $\overrightarrow{\chi_h}(D) \geq \lceil \frac{1+\sqrt{4m+1}}{2} \rceil$, where m is the number of arcs of D.*

Let G be an undirected graph. For any orientation O of the edges of G, $G(O)$ is a directed graph. Since Theorem 3.1.1 is true for $G(O)$ for any arbitrary orientation O, the following result holds good.

Corollary 3.1.2. *Let G be any undirected graph. Then $\overrightarrow{\chi_h}(O(G)) \geq \lceil \frac{1+\sqrt{4m+1}}{2} \rceil$, where m is the number of edges of G.*

3.2 Lower bound for $\overrightarrow{\chi_h}$ of regular digraphs

In this section, we obtain a lower bound for proper harmonious coloring number of regular digraphs. We have already determined our first lower bound for digraphs in Chapter 2. We now give another lower bound for $\overrightarrow{\chi_h}(D)$, where D is a regular digraph.

Theorem 3.2.1. *Let D be an r- regular digraph of order n. Then $\overrightarrow{\chi_h}(D) \geq \lceil \frac{1+\sqrt{1+4rn}}{2} \rceil$.*

Proof. Let D be harmoniously colored with k colors. There exists at least one color class, say X, that contains at least $\frac{n}{k}$ vertices. For, if every color class contains less than $\frac{n}{k}$ vertices, then the number of vertices in D is less than n, which is a contradiction. Let $N(X)$ denote the neighbourhood of the color class X. Since D is regular of degree r and no two vertices in the same color class have a common inneighbour or outneighbour, it follows that there are at least $r(\frac{n}{k})$ vertices in $N(X)$. Each of these vertices must be assigned a distinct color. Thus the total number of

colors is $k \geq |N(X)| + 1$.

This implies $k^2 - k - rn \geq 0$.

Therefore, $(k - (\frac{1+\sqrt{1+4rn}}{2}))(k - (\frac{1-\sqrt{1+4rn}}{2})) \geq 0$

But $(k - (\frac{1-\sqrt{1+4rn}}{2})) > 0$ and hence $(k - (\frac{1+\sqrt{1+4rn}}{2})) \geq 0$

Thus, $\overrightarrow{\chi_h}(D) \geq \lceil \frac{1+\sqrt{1+4rn}}{2} \rceil$.

$\square$

Remark 3.2.2. *When* $r = 1$, $\overrightarrow{\chi_h}(D) \geq \lceil \frac{1+\sqrt{1+4n}}{2} \rceil$.

But $r = 1$ *implies that* D *is the unicycle* $\overrightarrow{C_n}$ *with* n *vertices. Also, in Chapter 2, we have already proved that for* $k = \lceil \frac{1+\sqrt{4n+1}}{2} \rceil$,

$$\overrightarrow{\chi_h}(\overrightarrow{C}_n) = \begin{cases} k+1 & if \quad n = k(k-1) - 1, \\ k & if \quad n = (k-1)(k-2) + 1, ..., k(k-1) - 2, k(k-1). \end{cases}$$

Let G be an undirected graph. Consider all orientations of edges of G such that $G(O)$ is regular. Since Theorem 3.2.1 is true for all orientations of G, we have the following result.

Corollary 3.2.3. *Let* G *be an undirected* $2r$-*regular graph. Then* $\overrightarrow{\chi_h}(O(G)) \geq \lceil \frac{1+\sqrt{1+4rn}}{2} \rceil$.

In the next sections, we determine the proper harmonious coloring number of some classes of regular digraphs such as Torus and Circulant digraphs.

3.3   Proper harmonious coloring number of Torus

Definition 3.3.1. *An n-dimensional torus is defined as an interconnection structure that has $k_0 \times k_1 \times \cdots \times k_{n-1}$ nodes, where k_i is the number of nodes in i^{th} dimension. Each node in the torus is identified by an n-coordinate vector $(x_0, x_1, \cdots, x_{n-1})$, where $0 \leq x_i \leq k_i - 1$. Two nodes $(x_0, x_1, \cdots, x_{n-1})$ and $(y_0, y_1, \cdots, y_{n-1})$ are connected if and only if there exists an i such that $x_i = (y_i \pm 1) \mod k_i$ and $x_j = y_j$ for all $j \neq i$.*

Theorem 3.3.2. *Let T be an undirected $n \times n$ torus. Then $\overrightarrow{\chi_h}(O(T)) = k + 1$, where $k = \lceil \frac{1+\sqrt{1+8n^2}}{2} \rceil$.*

Proof. By Theorem 3.2.1, $\overrightarrow{\chi_h}(O(T)) \geq k = \lceil \frac{1+\sqrt{1+8n^2}}{2} \rceil$.

Let O be the orientation of T such that all the horizontal cycles are unidirected in the clockwise sense and all the vertical cycles are unidirected in the clockwise sense to obtain the orientation $O(T)$ (See Figure 3.1).
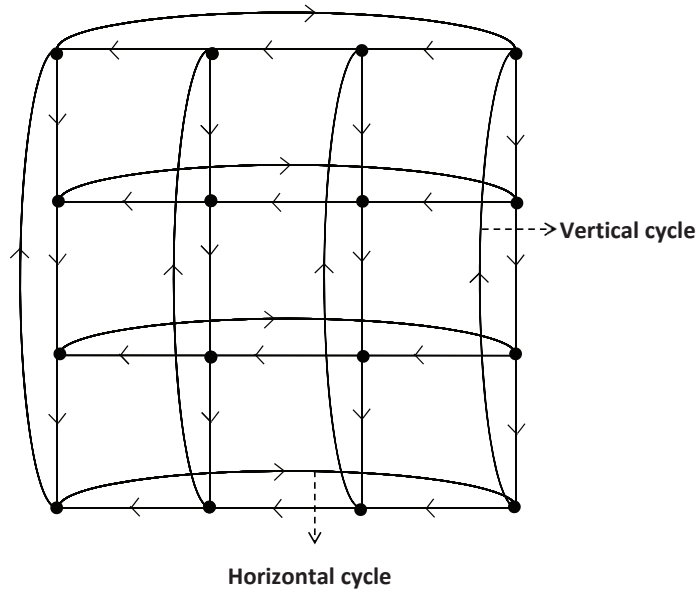
Figure 3.1:   Orientation $O(T)$ for 4×4 torus.

Consider a complete symmetric digraph $\overleftrightarrow{K}_{k+1}$ with $k(k+1)$ arcs. Label the vertices of $\overleftrightarrow{K}_{k+1}$ as $1, 2, ..., k+1$. Let $v_1, v_2, v_3, ..., v_{n(n+1)}, v_1$ be a closed trail $\overrightarrow{W}$ of length $n(n+1)$ in $\overleftrightarrow{K}_{k+1}$. Then to find the proper harmonious coloring of $O(T)$, it is sufficient to find a closed trail $\overrightarrow{W}$ of length $n(n+1)$ in $\overleftrightarrow{K}_{k+1}$ traversing through the arcs of $\overleftrightarrow{K}_{k+1}$ at most once

(For example, consider $\overleftrightarrow{K}_6$, complete symmetric digraph with 6 vertices. Let the closed trail be $1-2-3-5-3-4-5-4-2-6-4-3-1$. Now, this closed trail can be used to find the proper harmonious coloring of an oriented Torus$(3,3)$(See Figure 3.2).)
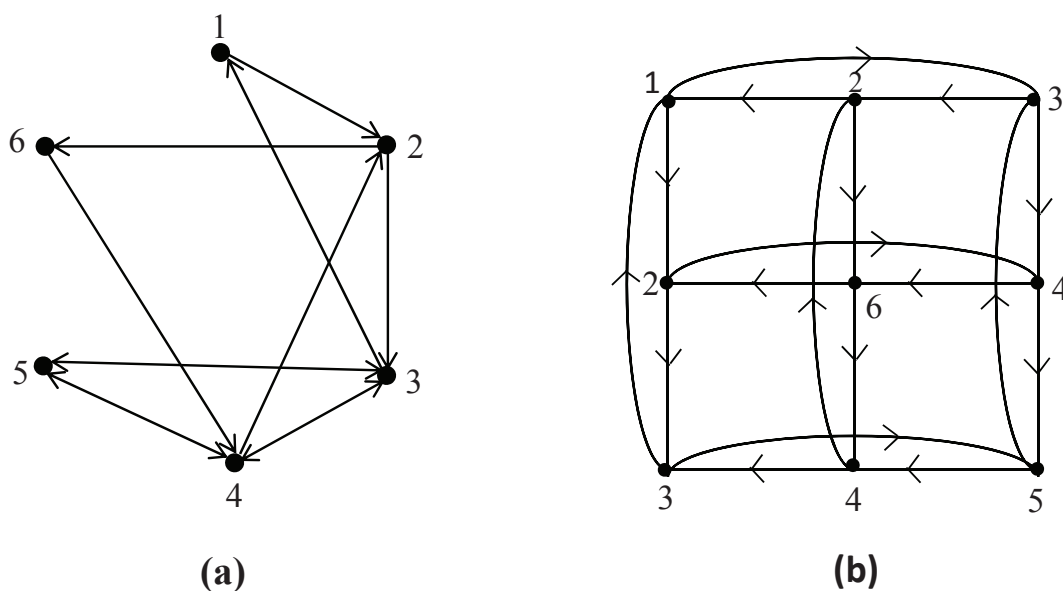
Figure 3.2: (a) Closed trail $\overrightarrow{W}$ in $\overleftrightarrow{K}_6$; (b) Oriented Torus$(3,3)$.

with the following conditions:

- In $\overrightarrow{W}$, the label of the vertex $v_2=$ the label of the vertex $v_{n(n-1)+n}$, the label of the vertex $v_3=$ the label of the vertex $v_{n(n-2)+(n-1)}$, the label of the vertex $v_4=$ the label of the vertex $v_{n(n-3)+(n-2)}$, ..., the label of the vertex $v_n=$ the label of the vertex v_{n+2}.

- In $\overrightarrow{W}$, the arcs (v_{n+2}, v_{2n+3}); (v_{2n+3}, v_{3n+4}); (v_{3n+4}, v_{4n+5});...;(v_{n^2}, v_1); (v_{n+3}, v_{2n+4});

$(v_{2n+4}, v_{3n+5}); ...; (v_{n^2+1}, v_2); (v_{n+4}, v_{2n+5}); (v_{2n+5}, v_{3n+6}); ...; (v_{n^2+2}, v_3); ...; (v_{2n}, v_{3n+1}); (v_{3n+1}, v_{4n+2}); (v_{4n+2}, v_{5n+3}); ...; (v_{n^2+(n-2)}, v_{n-1})$ do not exist.

- Also in $\overrightarrow{W}$, the arcs $(v_1, v_{n+2}); (v_2, v_{n+3}); (v_3, v_{n+4}); ...; (v_{n-1}, v_{2n})$ do not exist. (See Figure 3.3)

Figure 3.3: Oriented Torus(n, n).

There exists at least one such closed trail in $\overleftrightarrow{K}_{k+1}$ satisfying the above conditions. Further, $\overrightarrow{\chi_h}(O(T)) \neq k$ as the number of ordered pairs obtained from these k colors are not sufficient to label any $n \times n$ grid $O(T)$ with distinct ordered pairs. Thus $\overrightarrow{\chi_h}(O(T)) = k + 1$.

$\square$

3.4 Proper harmonious coloring number of Circulant digraph

The circulant is a natural generalization of the double loop net work and was first considered by Wong and Coppersmith (1974). Circulant graphs have been used for decades in the design of computer and telecommunication networks due to their optimal fault-tolerance and routing capabilities (Boesch and Wang 1985). It is also used in VLSI design and distributed computation (Bermond et al. 1995; Beivide et al. 1991; Wilkov 1972). Theoretical properties of circulant graphs have been studied extensively and surveyed by Bermond et al. (1995). Every circulant graph is a vertex transitive graph and a Cayley graph (Xu 2001). Most of the earlier research concentrated on using the circulant graphs to build interconnection networks for distributed and parallel systems (Bermond et al. 1995; Boesch and Wang 1985).

The following definition of a circulant digraph was given by Elspas and Turner (1970).

Definition 3.4.1. *A circulant digraph, denoted by $\overrightarrow{G_n}(S)$, where $S \subseteq \{1, 2, \cdots, n-1\}$, $n \geq 2$, is defined as a digraph consisting of the vertex set $V = \{0, 1, \cdots, n-1\}$ and the arc set $E = \{(i, j) : there\ is\ s \in S\ such\ that\ j - i \equiv s(mod\ n)\}$.*

The directed cycle $\{0, 1, 2, \cdots, n-1\}$ is called the *outer cycle.* All other cycles are called *inner cycles.*

The digraph shown in Figure 3.4 is $\overrightarrow{G_8}(1,3)$. It is clear that $\overrightarrow{G_n}(1)$ is the unicycle $\overrightarrow{C_n}$ and $\overrightarrow{G_n}(1, 2, \cdots, n-1)$ is a complete digraph $\overrightarrow{K_k}$.
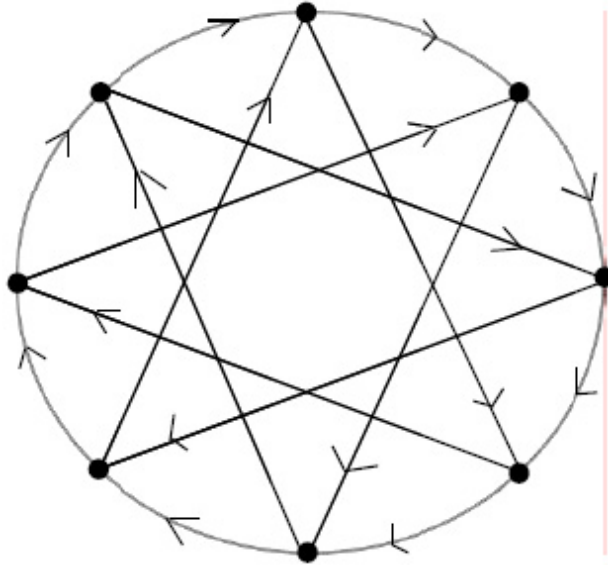
Figure 3.4: Circulant digraph $\overrightarrow{G_8}(1,3)$.

Theorem 3.4.2. *Let $\overrightarrow{G_n}(1,2)$ be a circulant digraph with n vertices. Then $k \leq$*

$\overrightarrow{\chi_h}(\overrightarrow{G_n}(1,2)) \leq k+1$ *for* $k = \lceil \frac{1+\sqrt{1+8n}}{2} \rceil$.

Proof. Clearly $\overrightarrow{G_n}(1,2)$ is 2-regular digraph. Then by Theorem 3.2.1, $\overrightarrow{\chi_h}(\overrightarrow{G_n}(1,2)) \geq$ $k = \lceil \frac{1+\sqrt{1+8n}}{2} \rceil$.

Let $\overleftrightarrow{K}_{k+1}$ be a complete symmetric digraph with $k(k+1)$ arcs. Label the vertices of $\overleftrightarrow{K}_{k+1}$ as $1, 2, ..., k+1$. To find the proper harmonious coloring of $\overrightarrow{G_n}(1,2)$, it is sufficient to find a closed trail, say, $\overrightarrow{W}$ of length $2n$ traversing through the arcs of $\overleftrightarrow{K}_{k+1}$ at most once.

(For example, consider $\overleftrightarrow{K}_6$, complete symmetric digraph with 6 vertices. Let the closed trail be $1-2-3-1-4-5-3-6-5-1-3-4-3-5-2-1-5-6-1$. Now, this closed trail can be used to find the proper harmonious coloring of a circulant digraph $\overrightarrow{G_9}(1,2)$. (See Figure 3.5))

Then there exists two cases:

Case (i) Let n be odd. Then in $\overrightarrow{G_n}(1,2)$, there exists only one inner cycle. Let $v_1, v_2, v_3, ..., v_n, v_1$ be any closed trail $\overrightarrow{W}$ of length n in $\overleftrightarrow{K}_{k+1}$ traversing through the
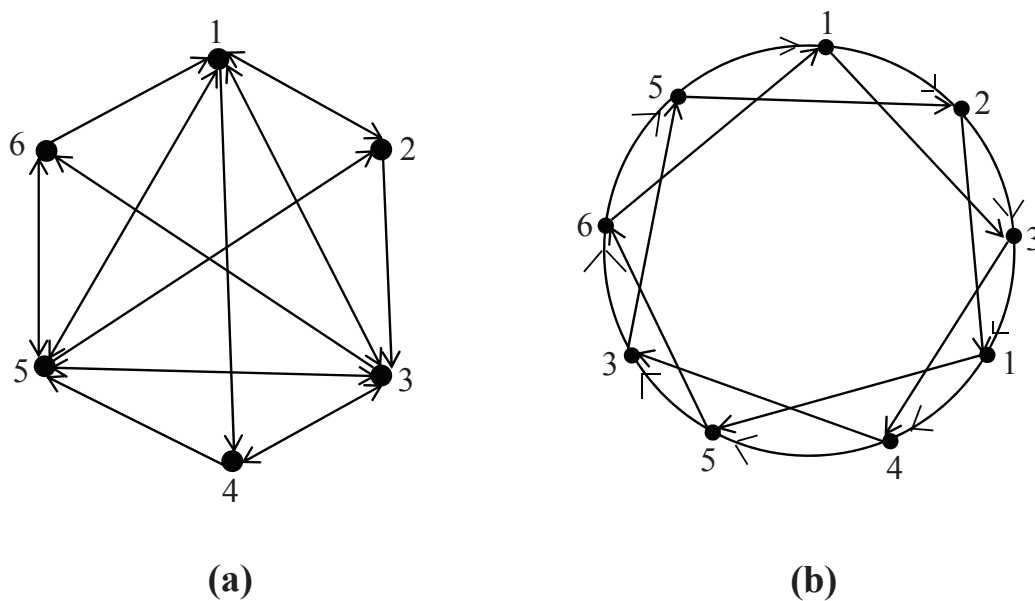
Figure 3.5: (a) Closed trail $\overrightarrow{W}$ in $\overleftrightarrow{K}_6$; (b) Circulant digraph $\overrightarrow{G_9}(1,2)$.

arcs of $\overleftrightarrow{K}_{k+1}$ at most once with the following conditions:

- If (v_i, v_j) and (v_j, v_k) are any two adjacent arcs in $\overrightarrow{W}$, then the arc (v_i, v_k) does not exist in $\overrightarrow{W}$ for any $i, j, k,\ 1 \le i \le n,\ 1 \le j \le n,\ 1 \le k \le n$. Also there shouldn't be any two arcs having arcs labels (v_i, v_k) in $\overrightarrow{W}$.

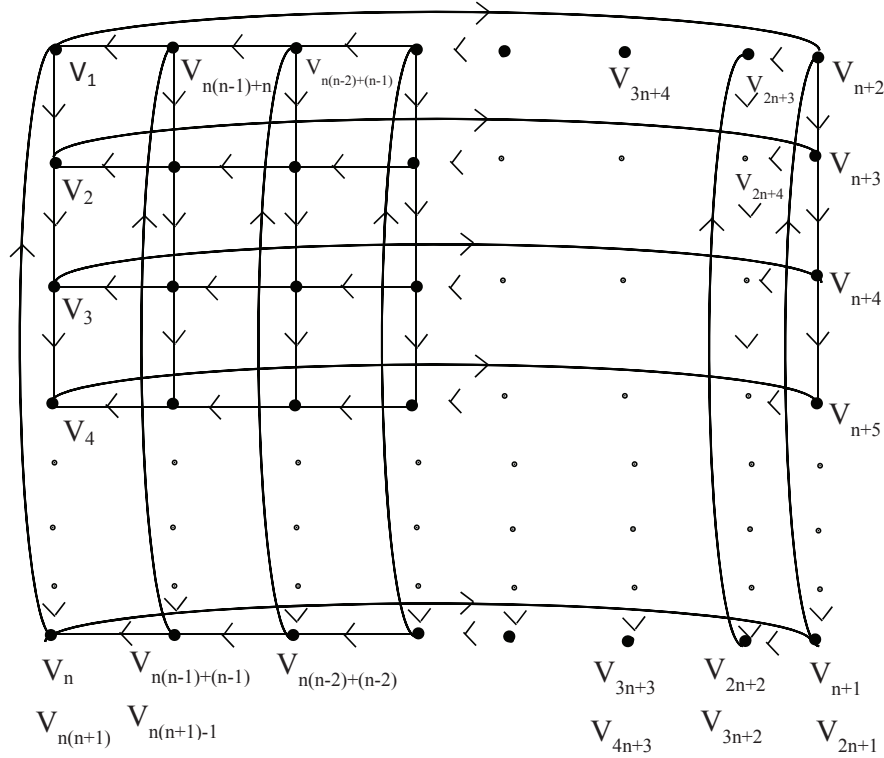- The arc (v_n, v_2) does not exist in the closed trail $\overrightarrow{W}$.(See Figure 3.6)

There exists at least one such closed trail in $\overleftrightarrow{K}_{k+1}$ satisfying the above conditions.
Case (ii) Let n be even. Then in $\overrightarrow{G_n}(1,2)$, there exist two inner cycles. Let
$v = v_1, v_2, v_3, ..., v_{n-1}, v_n = v, v_{n+1}, v_{n+2}, ..., v_{\frac{3n}{2}-1}, v_{\frac{3n}{2}}, v_{\frac{3n}{2}+1}, ..., v_{2n-1}, v_{2n}, v_{2n+1} = v$
be any closed trail $\overrightarrow{W}$ of length $2n$ in $\overleftrightarrow{K}_{k+1}$ traversing through the arcs of $\overleftrightarrow{K}_{k+1}$ at most once with the following condition:

- In $\overrightarrow{W}$, the label of the vertex $v_1 =$ the label of the vertex $v_n =$ the label of the vertex v_{2n+1}, the label of the vertex $v_3 =$ the label of the vertex v_{n+1}, the label of the vertex $v_5 =$ the label of the vertex v_{n+2}, ..., the label of the vertex $v_{n-1} =$ the label of the vertex $v_{\frac{3n}{2}-1}$, the label of the vertex $v_2 =$ the label of the vertex
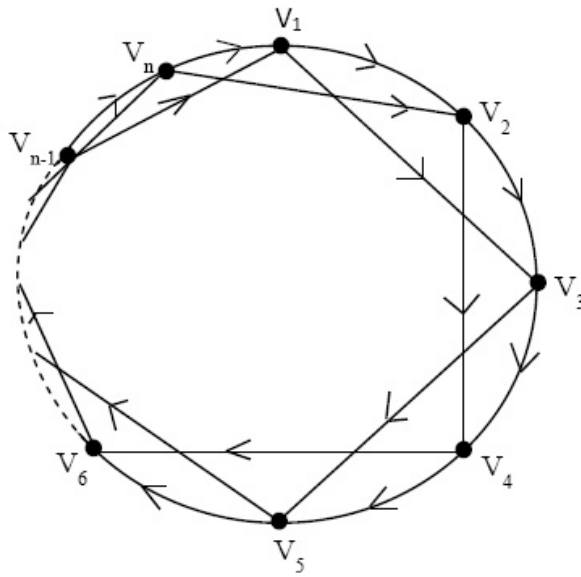
Figure 3.6: Circulant digraph $\overrightarrow{G_n}(1,2)$.

$v_{\frac{3n}{2}+1}$, the label of the vertex $v_4 =$ the label of the vertex $v_{\frac{3n}{2}+2}$, the label of the vertex $v_6 =$ the label of the vertex $v_{\frac{3n}{2}+3}$, ..., the label of the vertex $v_{\frac{3n}{2}} =$ the label of the vertex v_{2n}.(See Figure 3.7)

There exists at least one such closed trail in $\overleftrightarrow{K}_{k+1}$ satisfying the above condition.

$\square$
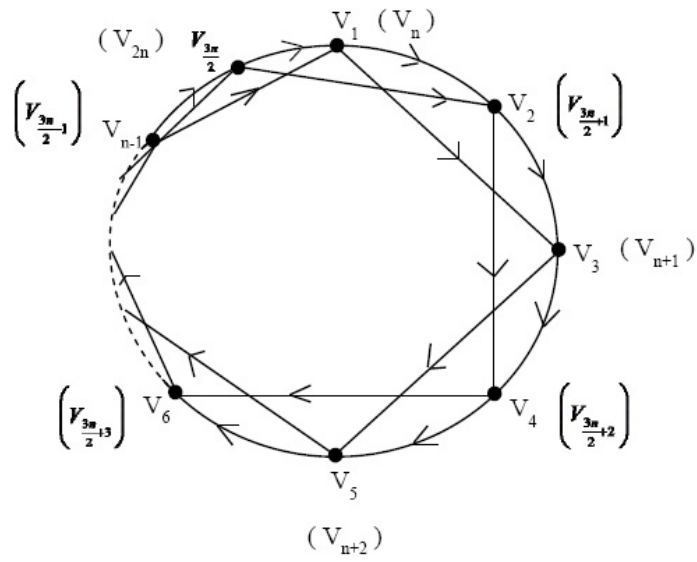
Figure 3.7: Circulant digraph $\overrightarrow{G_n}(1,2)$.

Chapter 4

Complete Colorings of Digraphs

The proper vertex coloring of a graph G in which we are most interested are those use the smallest number of colors. In graph theory, complete coloring is the opposite of harmonious coloring in the sense that it is a vertex coloring in which every pair of colors appears on at least one pair of adjacent vertices. The achromatic number of a graph G is the maximum number of colors possible in any complete coloring of G.

In this chapter we focus on a type of vertex coloring called complete colorings of digraphs. Also, we obtain an upper bound for the achromatic number of digraphs for which complete coloring is possible and investigate the same for different types of digraphs.

4.1   Introduction

The following is an extension of complete colorings to directed graphs.

Definition 4.1.1. *Let D be a directed graph with n vertices and m arcs. A function $g : V(D) \to \{1, 2, ..., k\}$, where $k \leq n$ is called a complete coloring of D if and only if for each ordered pair (c, c') of colors, there is an arc (u, v) of D such that $(g(u), g(v)) = (c, c')$. If the pair (i, i) is not assigned, then g is called a proper complete*

coloring of D. The maximum k for which D admits a proper complete coloring is called the achromatic number of D and is denoted by $\overrightarrow{\psi}(D)$.

In Figure 4.1, complete coloring of oriented Petersen graph is displayed.
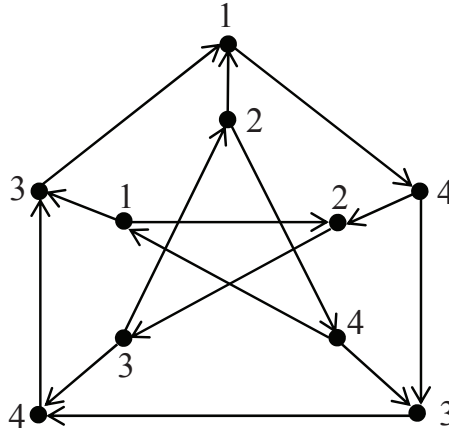
Figure 4.1: A proper complete coloring of oriented Petersen graph.

Example:

(a)   **(b)**
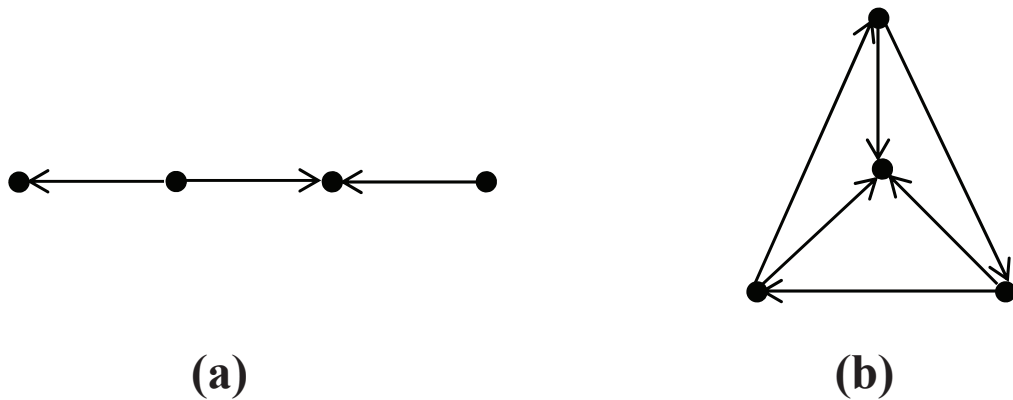
Figure 4.2: (a) Oriented path; (b) Unicyclic wheel $\overrightarrow{W_4}$.

From Figure 4.2, one can see that not every digraph has a proper complete coloring and hence it is interesting to investigate the complete coloring of digraphs.

In the next section, we give an upper bound for the achromatic number of digraphs for which complete coloring is possible.

4.2 Upper bound for the achromatic number of digraphs

The following theorem gives an upper bound for $\overrightarrow{\psi}(D)$.

Theorem 4.2.1. *If a digraph D has a proper complete coloring, then $\overrightarrow{\psi}(D) \leq \lfloor \frac{1+\sqrt{4m+1}}{2} \rfloor$, where m is the number of arcs of D.*

Proof. Let D be a digraph having m arcs with proper complete coloring using k colors. We know that the possible number of ordered pairs is $k(k-1)$.

$\therefore m \geq k(k-1)$.

$\Rightarrow k^2 - k - m \leq 0$.

$\Rightarrow k \leq \frac{1+\sqrt{4m+1}}{2}$.

$\therefore k \leq \lfloor \frac{1+\sqrt{4m+1}}{2} \rfloor$.

$\square$

The following corollary gives an upper bound for the achromatic number of regular digraphs.

Corollary 4.2.2. *Let D be an r- regular digraph of order n. Then $\overrightarrow{\psi}(D) \leq \lfloor \frac{1+\sqrt{1+4rn}}{2} \rfloor$.*

Proof. Let D be proper completely colored with k colors. Since D is r - regular, it has $m = rn$ arcs. Then from Theorem 4.2.1, it follows that

$\overrightarrow{\psi}(D) \leq \lfloor \frac{1+\sqrt{1+4rn}}{2} \rfloor$.

$\square$

In general, the proper complete coloring problem has been viewed as an Eulerian path decomposition of digraphs. Whenever such a decomposition possible, it is possible to find the proper complete coloring of digraphs.

Let $D(G)$ be a symmetric digraph of any undirected graph G. Then to find a proper complete coloring of any digraph D with n vertices, it is sufficient to find a

closed walk traversing through all the arcs at least once of length n in $D(G)$ and the number of vertices of $D(G)$ gives the achromatic number of D.

Example:

Let $D(G) = \overleftrightarrow{K}_5$, a complete symmetric digraph with 5 vertices.
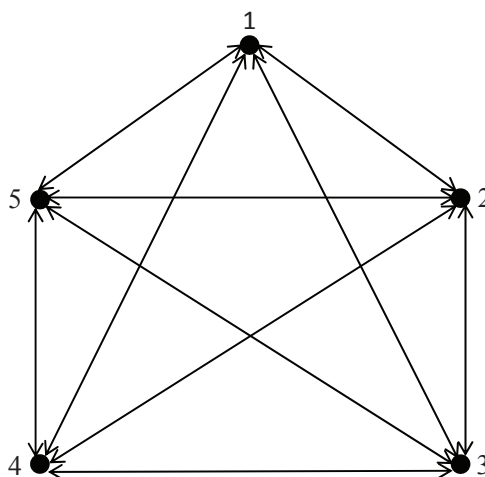
Figure 4.3: Complete symmetric digraph $\overleftrightarrow{K_5}$.

Let the closed walk be $1 - 2 - 3 - 4 - 5 - 1 - 3 - 5 - 2 - 4 - 1 - 5 - 4 - 3 - 2 - 1 - 4 - 2 - 5 - 3 - 1$. This closed walk can be used to find the proper complete coloring of the unipath $\overrightarrow{P}_{21}$ with 21 vertices and unicycle $\overrightarrow{C}_{20}$ with 20 vertices. (See Figure 4.4)

Thus, to find a proper complete coloring of any digraph D with n vertices, it is sufficient to find a closed walk traversing through all the arcs at least once of length n in $D(G)$ and the number of vertices of $D(G)$ gives the achromatic number of D.

In the next section, we determine the achromatic number of some familiar classes of digraphs.

(a)

(b)
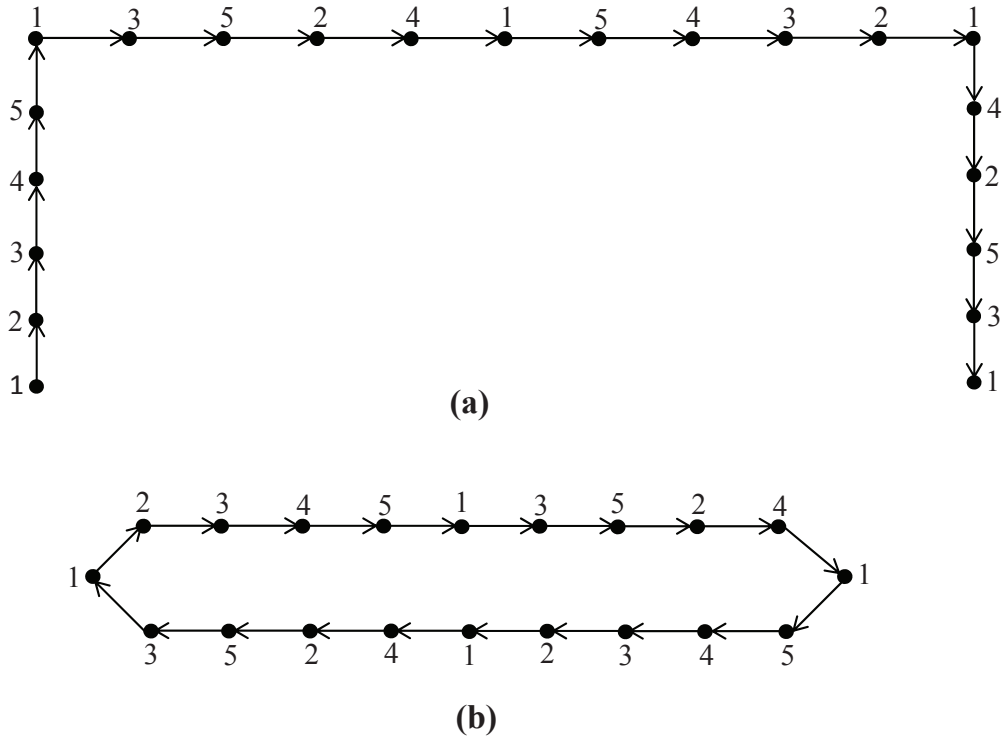
Figure 4.4: A proper complete coloring of (a) $\overrightarrow{P_{21}}$; (b) $\overrightarrow{C_{20}}$.

4.3 Achromatic number of some classes of digraphs

Definition 4.3.1. *Let D_1 and D_2 be any digraphs. A **homomorphism** (Hell and Nesetril 2004) of D_1 to D_2, written as $f : D_1 \to D_2$ is a mapping $f : V(D_1) \to V(D_2)$ such that $(f(u), f(v)) \in E(D_2)$ whenever $(u, v) \in E(D_1)$.*

We first consider the achromatic number of a unipath $\overrightarrow{P}_n$ on n vertices.

Theorem 4.3.2. *Let $\overrightarrow{P_n}$ be a unipath with n vertices, where $n \geq 3$, then $\overrightarrow{\psi}(\overrightarrow{P_n}) = \lfloor \frac{1+\sqrt{4(n-1)+1}}{2} \rfloor$.*

Proof. Since unipath $\overrightarrow{P}_n$ contains $(n-1)$ arcs, by Theorem 4.2.1, $\overrightarrow{\psi}(\overrightarrow{P}_n) \leq \lfloor \frac{1+\sqrt{4(n-1)+1}}{2} \rfloor$. Let $k = \lfloor \frac{1+\sqrt{4(n-1)+1}}{2} \rfloor$. Then, $k(k-1)+1 \leq n < k(k+1)+1$. There are two inequalities to prove. One follows directly from Theorem 4.2.1, the

other from the fact that a complete symmetric digraph $\overleftrightarrow{K}_k$ on k vertices is $(k-1)$-regular and so has an Eulerian cycle. Thus, if $n - 1 \geq k(k-1)$, the path can be mapped homomorphically onto $\overleftrightarrow{K}_k$ in such a way that its arcs are mapped onto the Eulerian cycle. Indeed, if $\overrightarrow{P}_n = v_0, v_1, \cdots, v_{n-1}$ and if the Eulerian cycle is $u_0, u_1, \cdots, u_{k(k-1)-1}$, then the mapping $\phi : V(\overrightarrow{P}_n) \rightarrow V(\overleftrightarrow{K}_k)$ given by $\phi(v_i) = u_{i(mod\ k(k-1))}$ gives the proper complete coloring of the unipath $\overrightarrow{P}_n$.

$\square$

Figure 4.5 is an illustration of the above result.

$$1 \qquad 2 \qquad 3 \qquad 1 \qquad 3 \qquad 2 \qquad 1$$
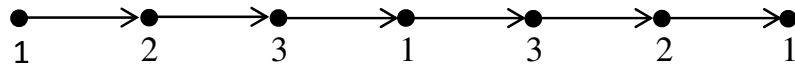
Figure 4.5: A proper complete coloring of $\overrightarrow{P}_7$.

Lemma 4.3.3. *If G is a connected (undirected) non-bipartite Eulerian graph, then the symmetric digraph $D(G)$ obtained from G has an alternating closed Eulerian trail.*

We have already proved the above lemma in Chapter 2.

Remark 4.3.4. *The lemma still follows if G is a connected (undirected) loopless multigraph.*

Next, we consider the achromatic number of an alternating path $\overrightarrow{A}P_n$ on n vertices.

Theorem 4.3.5. *Let $\overrightarrow{AP_n}$ be an alternating path with n vertices and $n \geq 7$. Then*

$$\overrightarrow{\psi}(\overrightarrow{AP_n}) = \begin{cases} k - 1 & \text{when } k \text{ is even and } k^2 - k + 1 \leq n \leq k^2 - 1 \\ k & \text{otherwise,} \end{cases}$$

where $k = \lfloor \frac{1 + \sqrt{1 + 4(n-1)}}{2} \rfloor$.

Proof. Since $\overrightarrow{AP_n}$ is an alternating path with n vertices and $n-1$ arcs, by Theorem 4.2.1, $\overrightarrow{\psi}(\overrightarrow{AP_n}) \leq \lfloor \frac{1+\sqrt{1+4(n-1)}}{2} \rfloor$. Let $k = \lfloor \frac{1+\sqrt{1+4(n-1)}}{2} \rfloor$. Then $k(k-1)+1 \leq n \leq k(k+1)$.

Let G be a connected (undirected) non-bipartite graph with all degrees even. Then to find the proper complete coloring of $\overrightarrow{AP_n}$, it is sufficient to find an alternating Eulerian trail in $D(G)$ of length n and the number of vertices of $D(G)$ gives the achromatic number of $\overrightarrow{AP_n}$. Consider a complete undirected graph K_k with k vertices.

Case (i) Let k be odd and let $G = K_k$. Then G contains $m = \frac{k(k-1)}{2}$ edges. Since G is non-bipartite and all the vertices are of even degree, G has an undirected closed Eulerian trail T. Then by Lemma 4.3.3, we obtain the required alternating closed Eulerian trail in $D(G)$.

Case (ii) Let k be even and let G be a graph obtained by adding a matching of size $k/2$ to K_k. Then G will have k vertices and $m = \frac{k^2}{2}$ edges. Also, all the vertices of G are of even degree. Hence G has an undirected closed Eulerian trail T. As m will be even for any value of k, by Lemma 4.3.3, we can find an alternating closed Eulerian trail $\overrightarrow{AT}$ in $D(G)$ such that the last arc of $\overrightarrow{AT}$ is the arc of the matching and the length of this alternating closed Eulerian trail $\overrightarrow{AT}$ is $2(\binom{k}{2} + k/2) = k^2$. Since we don't require alternating closed Eulerian trail, we can remove the last arc of $\overrightarrow{AT}$ (since it is the arc of the matching and as it is repeated) so that we obtain the required alternating Eulerian trail of length $k^2 - 1$.

As k colors are used to color the vertices of $\overrightarrow{AP_n}$ of length $n-1$, there are $k(k-1)$ ordered pairs of colors. In $\overrightarrow{AP_n}$, at each vertex there will be either two incoming arcs or two outgoing arcs except for the first and the last vertex. Hence there should be even number of ordered pairs having one particular color. There are $k-1$ ordered pairs associated with each color. When k is even, $k-1$ will be odd and hence in order to use all the ordered pairs of one particular color, we have to repeat some ordered pairs. Hence for k colors, we must repeat $k-1$ ordered pairs (since first and last vertex doesn't have two incoming or two outgoing arcs). Thus, when k is even, for

an alternating path with less than $k^2 - 1$ vertices, we require $k - 1$ colors. Hence $\overrightarrow{\psi}(\overrightarrow{AP_n}) = k - 1$ when k is even and $k^2 - k + 1 \leq n \leq k^2 - 1$.

□

Figures 4.6, 4.7 and 4.8 display an example of a proper complete coloring of $\overrightarrow{AP}_{21}$, $\overrightarrow{AP}_{16}$ and $\overrightarrow{AP}_{13}$ respectively.

Figure 4.6: A proper complete coloring of $\overrightarrow{AP}_{21}$.

Figure 4.7: A proper complete coloring of $\overrightarrow{AP}_{16}$.
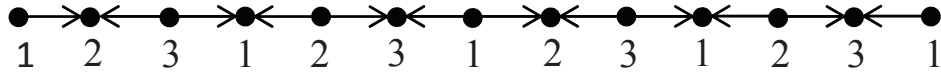
Figure 4.8: A proper complete coloring of $\overrightarrow{AP}_{13}$.

Theorem 4.3.6. *Let $\overrightarrow{C}_n$ be a unicycle with n vertices, where $n \neq 3$, 5 and 7. Then, for $k = \lfloor \frac{1+\sqrt{4n+1}}{2} \rfloor$, $k \geq 3$,*

$$
\overrightarrow{\psi}(\overrightarrow{C}_n) = \begin{cases} 2 & if \ \ n = 4 \\[2mm] k & for \ \ n = k(k-1), k(k-1)+2, k(k-1)+3, ..., k(k+1) - 1 \\[2mm] k - 1 & for \ \ n = k(k-1)+1, \ n \neq 7. \end{cases}
$$

Proof. When $n = 3$, 5, 7, there exists at least one arc in $\overrightarrow{C}_n$ for which the ordered pair (i, i) appears. Thus it is not possible to color the vertices with proper complete coloring in these cases.

One can easily verify that $\overrightarrow{\psi}(\overrightarrow{C}_n) = 2$ if $n = 4$.

Since a unicycle $\overrightarrow{C}_n$ contains n arcs, by Theorem 4.2.1, $\overrightarrow{\psi}(\overrightarrow{C_n}) = k \leq \lfloor \frac{1+\sqrt{4n+1}}{2} \rfloor$. Let $k = \lfloor \frac{1+\sqrt{4n+1}}{2} \rfloor$. Then $k(k-1) \leq n \leq k(k+1) - 1$. To prove the result, it is enough to show that there exists a closed walk of length at least n traversing through all the arcs at least once in $\overleftrightarrow{K}_k$, for $k(k-1) \leq n \leq k(k+1) - 1$. In $\overleftrightarrow{K}_k$, $id(v) = od(v) = k - 1$ for all v. Hence $\overleftrightarrow{K}_k$ is Eulerian. Hence we can find a closed walk of length $k(k-1)$ and the number of vertices of $\overleftrightarrow{K}_k$ gives the achromatic number. Thus $\overrightarrow{\psi}(\overrightarrow{C}_n) = k$ for $n = k(k-1)$. Then starting from the end vertex of the closed walk obtained in $\overleftrightarrow{K}_k$ as above, traverse through $2, 3, 4, ..., 2k - 1$ arcs which makes a closed walk with the starting point and is of length $k(k-1) + 2, k(k-1) + 3, ..., k(k+1) - 1$. Hence $\overrightarrow{\psi}(\overrightarrow{C}_n) = k$ for $n = k(k-1) + 2, \; k(k-1) + 3, \cdots, \; k(k+1) - 1$.

When $n = k(k-1) + 1$, we cannot find a closed walk of length n in $\overleftrightarrow{K}_k$. For, if it is possible to find a closed walk of length n, then for k vertices, the indegree sequence of arcs is equal to the outdegree sequence of the arcs which is equal to $(k - 1, k - 1, k - 1, ..., k - 1, k)$. But there exists no such graph. Hence when $n = k(k-1)+1$, we require only $k-1$ colors. That is, $\overrightarrow{\psi}(\overrightarrow{C}_n) = k-1$ for $n = k(k-1)+1$.

$\square$

Figures 4.9 and 4.10 are the illustrative examples of the above result.
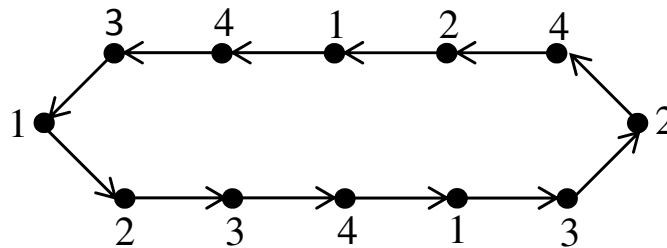
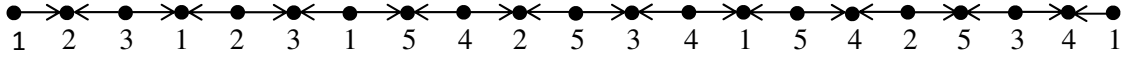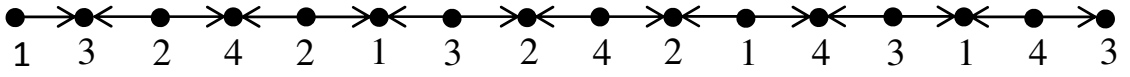Figure 4.9: A proper complete coloring of $\overrightarrow{C}_{12}$.
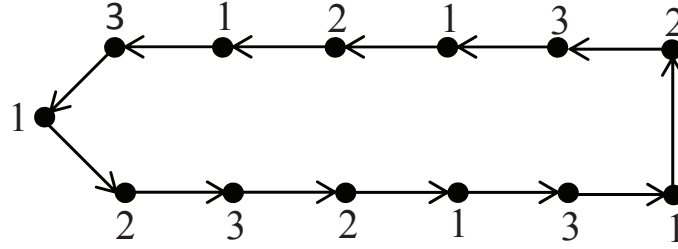
Figure 4.10: A proper complete coloring of $\overrightarrow{C_{13}}$.

**Theorem 4.3.7.** *Let* $\overrightarrow{AC_n}$ *be an alternating cycle with n vertices, where n is even and $n \geq 6$. Then*

$$\overrightarrow{\psi}(\overrightarrow{AC_n}) = \begin{cases} k-1 & \text{when k is even and $k(k-1) \leq n \leq k^2 - 2$} \\ k & \text{otherwise,} \end{cases}$$

where $k = \lfloor \frac{1+\sqrt{1+4n}}{2} \rfloor$.

Proof. Since $\overrightarrow{AC_n}$ is an alternating cycle with n vertices and n arcs, by Theorem 4.2.1, $\overrightarrow{\psi}(\overrightarrow{AC_n}) \leq \lfloor \frac{1+\sqrt{1+4n}}{2} \rfloor$. Let $k = \lfloor \frac{1+\sqrt{1+4n}}{2} \rfloor$. Then $k(k-1) \leq n \leq k(k+1) - 2$. Let G be a connected (undirected) non-bipartite Eulerian graph. Then to find the proper complete coloring of $\overrightarrow{AC_n}$, it is sufficient to find an alternating closed Eulerian trail in $D(G)$ of length n and the number of vertices of $D(G)$ gives the achromatic number of $\overrightarrow{AP_n}$. Consider a complete undirected graph K_k with k vertices.

Case (i) Let k be odd and $G = K_k$. Then G contains $m = \frac{k(k-1)}{2}$ edges. Since G is non-bipartite and all the vertices are of even degree, G has an undirected closed Eulerian trail T. Then by Lemma 4.3.3, we obtain the required alternating closed Eulerian trail in $D(G)$.

Case (ii) Let k be even and let G be a graph obtained by adding a matching of size $k/2$ to K_k. Then G will have k vertices and $m = \frac{k^2}{2}$ edges. Also, all the vertices of G are of even degree. Hence G has an undirected closed Eulerian trail T. As m

will be even for any value of k, by Lemma 4.3.3, we can find an alternating closed Eulerian trail in $D(G)$ and the length of this alternating closed Eulerian trail is $2\left(\binom{k}{2} + k/2\right) = k^2$.

As k colors are used to color the vertices of $\overrightarrow{AC_n}$ of length n, there are $k(k-1)$ ordered pairs of colors. In $\overrightarrow{AC_n}$, at each vertex there will be either two incoming arcs or two outgoing arcs. Hence there should be even number of ordered pairs of one particular color. There are $k-1$ ordered pairs associated with each color. When k is even, $k-1$ will be odd and hence in order to use all the ordered pairs of one particular color, we have to repeat some ordered pairs. Hence for k colors, we must repeat k ordered pairs. Thus, when k is even, for an alternating cycle with less than k^2 vertices, we require $k-1$ colors. Hence $\overrightarrow{\psi}(\overrightarrow{AC_n}) = k-1$ when k is even and $k(k-1) \leq n \leq k^2 - 2$.

$\square$

Figures 4.11, 4.12 and 4.13 are illustrative examples of the above result.
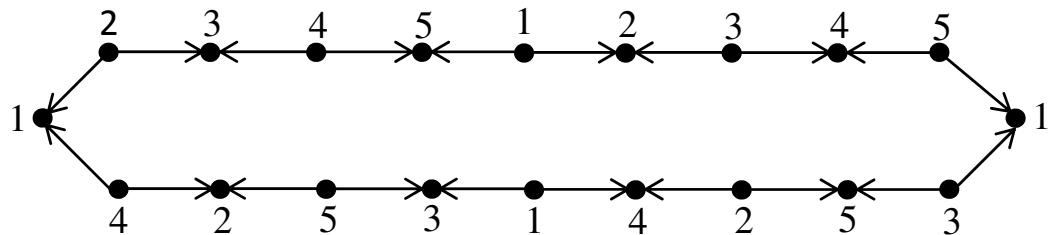
Figure 4.11: A proper complete coloring of $\overrightarrow{AC}_{20}$.
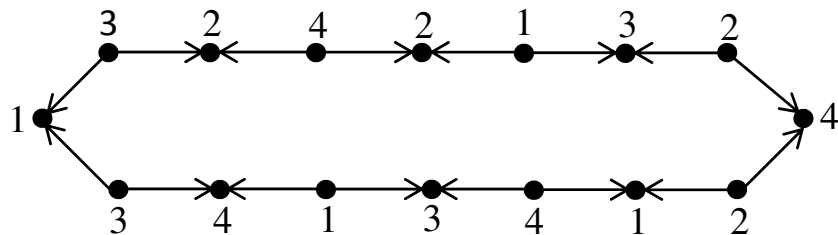
Figure 4.12: A proper complete coloring of $\overrightarrow{AC}_{16}$.
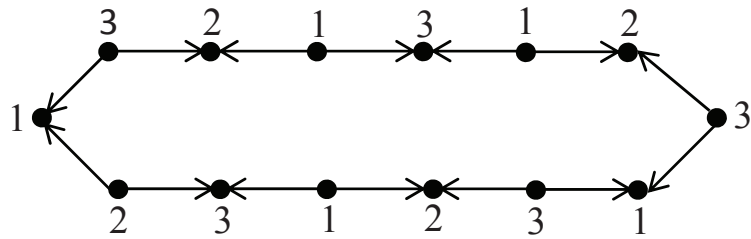
Figure 4.13: A proper complete coloring of $\overrightarrow{AC}_{14}$.

Theorem 4.3.8. *Let $\overrightarrow{S_n}$ be a directed star with n vertices. Then $\overrightarrow{\psi}(\overrightarrow{S_n}) = 2$ except when $id(v) = 0$ or $od(v) = 0$, where v is the central vertex.*

Proof. Let $\overrightarrow{S}_n$ be a directed star, where n is the number of vertices. The maximum number of colors in a complete coloring of a directed star is 2 since if the central vertex v of a directed star is colored i, at best the ordered pairs (i, j) and (j, i) can appear for some (many) j, but no other pairs. The maximum is clearly reached if and only if neither the indegree nor the outdegre is 0.

$\square$

Figure 4.14 is an illustration of the above result.
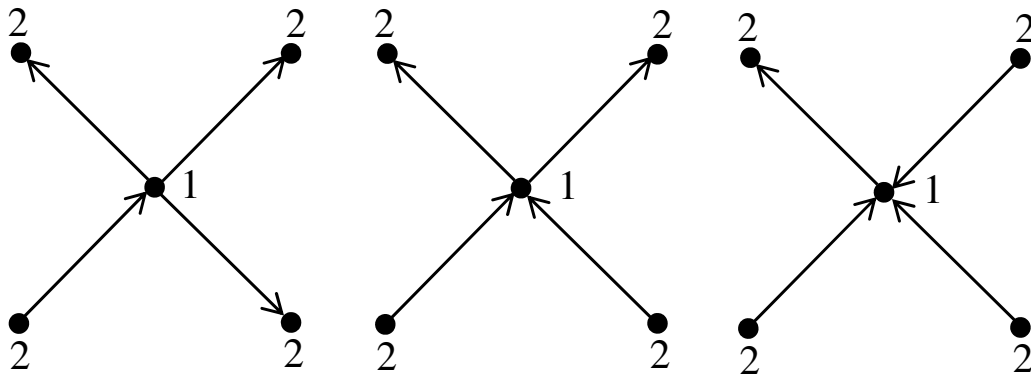
Figure 4.14: Proper complete coloring of $\overrightarrow{S_5}$.

Theorem 4.3.9. *Let $\overrightarrow{W_n}$ be a unicyclic wheel with odd number of vertices n and $id(v) = od(v)$ for the central vertex v. Then $\overrightarrow{\psi}(\overrightarrow{W_n}) = k$ for $k^2 - 3k + 3 \leq n \leq k^2 - k - 1$, where $k \geq 3$.*

Proof. Let $\overrightarrow{W_n}$ be a unicyclic wheel where $n \geq 5$ and n is odd (otherwise $id(v) \neq od(v)$, where v is the central vertex). The total number of arcs of the wheel is $2(n-1)$.

Case (i) Let $k = 3$. Then $n = 5$ and we can color the vertices of $\overrightarrow{W_5}$ as given below:
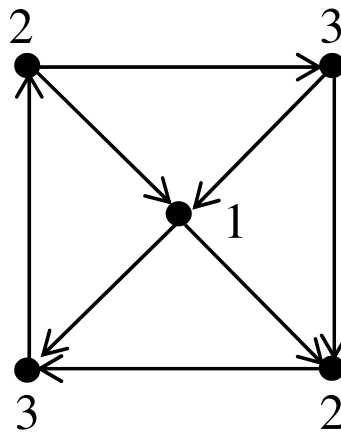
Figure 4.15: A proper complete coloring of $\overrightarrow{W_5}$.

Case (ii) Let $k > 3$. To find the proper complete coloring of the wheel $\overrightarrow{W_n}$, first we find the proper complete coloring of the rim. Consider a complete symmetric digraph $\overleftrightarrow{K}_{k-1}$. Then the proper complete coloring of the vertices of the rim of the wheel $\overrightarrow{W_n}$ is same as finding a closed walk (cycle) of length $n - 1$ traversing through all the arcs of $\overleftrightarrow{K}_{k-1}$ at least once. Since in $\overleftrightarrow{K}_{k-1}$, $id(v) = od(v) = k - 2$ for any vertex v, $\overleftrightarrow{K}_{k-1}$ is Eulerian. Hence we can find a closed walk of length $(n - 1)$ traversing through all the vertices of $\overleftrightarrow{K}_{k-1}$ when $n = k^2 - 3k + 3$. Then starting from the end vertex of the closed walk obtained in $\overleftrightarrow{K}_{k-1}$ as above, traverse through $2, 4, 6, ..., 2k-4$ arcs which makes a closed walk with the starting point and is of length $n - 1$ when $k^2 - 3k + 3 \leq n \leq k^2 - k - 1$. Hence we have a complete coloring of the rim of the wheel $\overrightarrow{W_n}$. Add one more vertex to $\overleftrightarrow{K}_{k-1}$. Then this vertex replaces the central vertex of the

wheel $\overrightarrow{W_n}$. Let $v_1, v_2, ..., v_{k-1}$ be the vertices of $\overleftrightarrow{K}_{k-1}$ and v_k be the additional vertex. Then by joining the vertices $v_1, v_2, ..., v_{k-1}$ of $\overleftrightarrow{K}_{k-1}$ with v_k in both the directions, we get $n-1$ ordered pairs (v_1, v_k), (v_2, v_k), ..., (v_{k-1}, v_k), (v_k, v_1), (v_k, v_2), ..., (v_k, v_{k-1}) when $n = k^2 - 3k + 3$. One can see that the inwardly and outwardly directed spokes are distributed around the wheel. For $k^2 - 3k + 3 < n \leq k^2 - k - 1$, repeat joining the vertices in both the directions to get the length as $(n-1)$. Hence by combining both, we get the complete coloring of the unicyclic wheel $\overrightarrow{W_n}$.

One can see that if we use an additional color, all the ordered pairs of colors cannot be assigned to the arcs of $\overrightarrow{W_n}$ to obtain a proper complete coloring. Hence it requires only k colors.

$\square$

Figure 4.16 is an illustration of the above result.
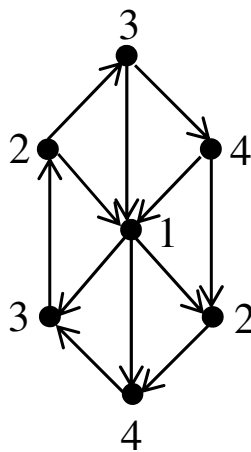
Figure 4.16: A proper complete coloring of $\overrightarrow{W_7}$.

Theorem 4.3.10. *For a complete binary out-tree $\overrightarrow{T}$ of level l, where l is even and $l = 4, 6, ...,$ then $\overrightarrow{\psi}(\overrightarrow{T}) = 2^{\frac{l}{2}} - 1$. (Note that the level of the tree $\overrightarrow{T}$ starts from 1,2,...)*

Proof. When $l = 2$, we cannot color the vertices of a binary out-tree $\overrightarrow{T}$ to obtain a complete coloring of $\overrightarrow{T}$. Also, we can observe that from level $2x$ to level $2x + 2$, $x = 2, 3, ...$, we require at most 2^x additional colors to obtain a complete coloring at level $2x + 2$. As l is even, let $l = 2x$, where $x = 2, 3, ...$ We shall prove the result by mathematical induction on x. When $x = 2$ ($l = 4$), as there are 14 arcs, we require only 3 colors for complete coloring. i.e. $\overrightarrow{\psi}(\overrightarrow{T}) = 2^{\frac{4}{2}} - 1 = 2^2 - 1 = 3$. Assume that the result is true for $x = m$. i.e. $\overrightarrow{\psi}(\overrightarrow{T}) = 2^{\frac{2m}{2}} - 1 = 2^m - 1$ for level $2m$. To prove that $\overrightarrow{\psi}(\overrightarrow{T}) = 2^{m+1} - 1$ for level $2(m + 1)$. We know that from level $2m$ to level $2(m + 1)$, it requires 2^m additional colors. Also, for level $2m$, by induction hypothesis, we use $2^m - 1$ colors. Hence for level $2(m + 1)$, $\overrightarrow{\psi}(\overrightarrow{T}) = 2^m - 1 + 2^m = 2.2^m - 1 = 2^{m+1} - 1$. Hence by the principle of mathematical induction, the result holds.

If we use one more additional color, all the ordered pairs of colors cannot be assigned to the arcs of $\overrightarrow{T}$ to obtain a proper complete coloring. Hence it requires only $2^{\frac{l}{2}} - 1$ colors.

$\square$

Figure 4.17 is an illustration of the above result.
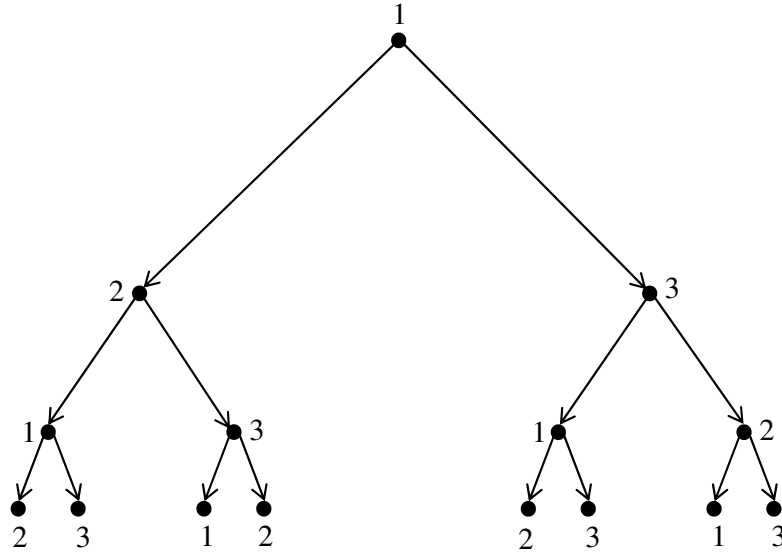
Figure 4.17: A proper complete coloring of $\overrightarrow{T}$ of level 4.

Circulant Digraphs

Next, we find the achromatic number of circulant digraphs $\overrightarrow{G_n}(1,2)$.

Theorem 4.3.11. *Let $\overrightarrow{G_n}(1,2)$ be a circulant digraph with n vertices, $n \geq 6$ and $n \neq 7$. Then for $k = \lfloor \frac{1+\sqrt{1+8n}}{2} \rfloor$, $k \geq 4$*

$$\overrightarrow{\psi}(\overrightarrow{G_n}(1,2)) = \begin{cases} k-1 & \text{when } k \text{ is odd and } n = \frac{k^2-k+2}{2} \\ k-1 & \text{when } k \text{ is even and } \frac{k(k-1)}{2} \leq n \leq \frac{k^2-2}{2}, \; n \neq 7 \\ k & \text{otherwise.} \end{cases}$$

Proof. When $n = 7$, it is not possible to find the achromatic number of $\overrightarrow{G_7}(1,2)$ as the arc (i,i) appears at least once.

Clearly $\overrightarrow{G_n}(1,2)$ is 2-regular digraph. Then by Corollary 4.2.2, $\overrightarrow{\psi}(\overrightarrow{G_n}(1,2)) \leq k = \lfloor \frac{1+\sqrt{1+8n}}{2} \rfloor$.

Let $\overleftrightarrow{K}_k$ be a complete symmetric digraph with $k(k-1)$ arcs. Label the vertices of $\overleftrightarrow{K}_k$ as $1, 2, ..., k$. To find the proper complete coloring of $\overrightarrow{G_n}(1,2)$, it is sufficient to

find a closed walk,say, $\overrightarrow{W}$ of length $2n$ traversing through all the arcs of $\overleftrightarrow{K}_k$ at least once.

(For example, consider $\overleftrightarrow{K}_4$, complete symmetric digraph with 4 vertices. Let the closed walk be $1-2-3-4-1-3-4-2-3-1-3-1-4-3-2-4-3-2-1$. Now, this closed walk can be used to find the proper complete coloring of a circulant digraph $\overrightarrow{G}_9(1,2)$. (See Figure 4.18))

(a)   (b)
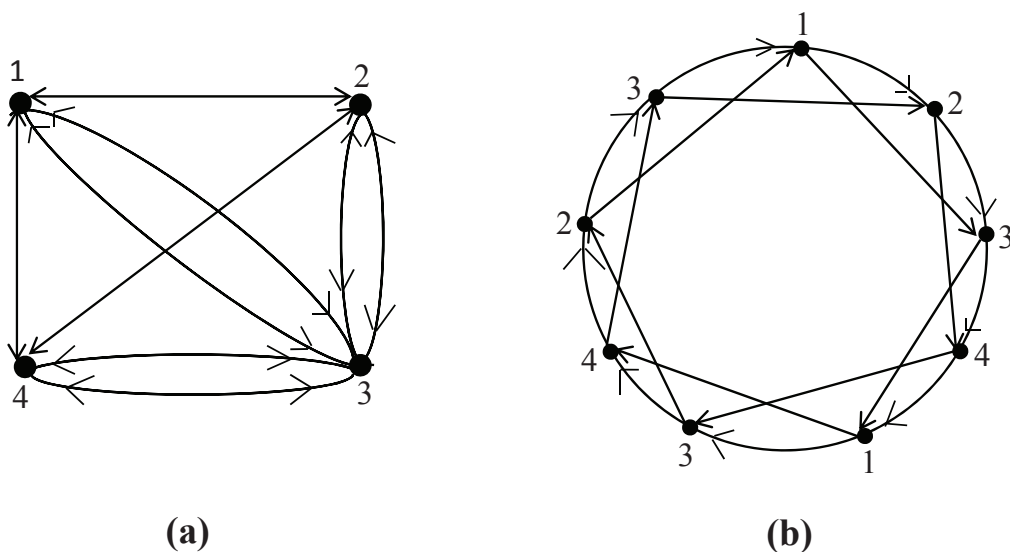
Figure 4.18: (a) Closed walk $\overrightarrow{W}$ in $\overleftrightarrow{K}_4$; (b) Circulant digraph $\overrightarrow{G}_9(1,2)$.

Let $v_1, v_2, v_3, ..., v_{n-1}, v_n$ be the vertices of $\overrightarrow{G}_n(1,2)$.

Case (i) Let n be odd. Then in $\overrightarrow{G}_n(1,2)$, let $v_1, v_2, v_3, ..., v_{n-1}, v_n, v_1$ be the outer cycle and there exists only one inner cycle, say, $v = v_{n+1}, v_{n+2}, ..., v_{2n}, v_{2n+1} = v$. Then one can see that $v = v_1, v_2, v_3, ..., v_{n-1}, v_n, v = v_{n+1}, v_{n+2}, ..., v_{2n}, v_{2n+1} = v$ is a closed walk $\overrightarrow{W}$ of length $2n$ in $\overleftrightarrow{K}_k$ traversing through all the arcs of $\overleftrightarrow{K}_k$ at least once with the following condition:

In $\overrightarrow{W}$, the label of the vertex $v_1=$ the label of the vertex $v_{n+1}=$ the label of the vertex v_{2n+1}, the label of the vertex $v_3=$ the label of the vertex v_{n+2}, the label of the vertex $v_5=$ the label of the vertex v_{n+3}, ..., the label of the vertex $v_n=$ the label of the vertex $v_{\frac{3n+1}{2}}$, the label of the vertex $v_2=$ the label of the vertex $v_{\frac{3n+3}{2}}$, the label of the vertex

$v_4=$ the label of the vertex $v_{\frac{3n+5}{2}}$, the label of the vertex $v_6=$ the label of the vertex $v_{\frac{3n+7}{2}}$, ..., the label of the vertex $v_{n-1}=$ the label of the vertex v_{2n} (See Figure 4.19). That is, there exists at least one such closed walk $\overrightarrow{W}$ in $\overleftrightarrow{K}_k$ satisfying the above condition.
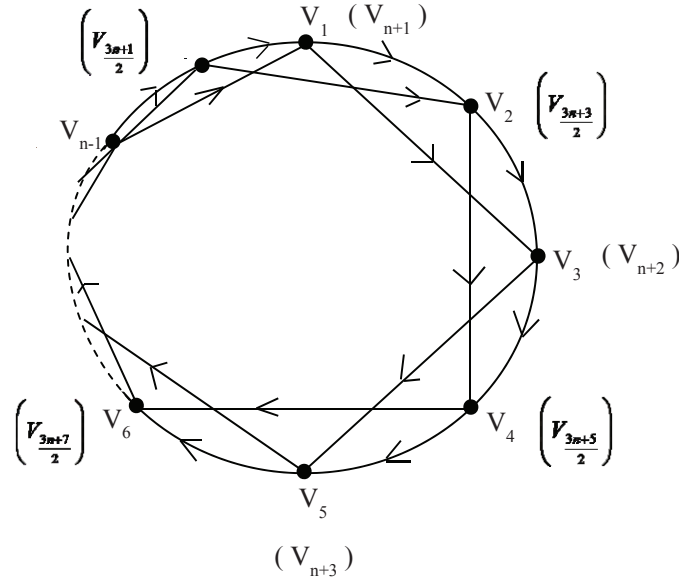
Figure 4.19: Circulant digraph $\overrightarrow{G_n}(1,2)$.

Case (ii) Let n be even. Then in $\overrightarrow{G_n}(1,2)$, there exist two inner cycles. Let
$$v = v_1, v_2, v_3, ..., v_{n-1}, v_n = v, v_{n+1}, v_{n+2}, ..., v_{\frac{3n}{2}-1}, v_{\frac{3n}{2}}, v_{\frac{3n}{2}+1}, ..., v_{2n-1}, v_{2n}, v_{2n+1} = v$$
be any closed walk $\overrightarrow{W}$ of length $2n$ in $\overleftrightarrow{K}_k$ traversing through all the arcs of $\overleftrightarrow{K}_k$ at least once with the following condition:

In $\overrightarrow{W}$, the label of the vertex $v_1=$ the label of the vertex $v_n=$ the label of the vertex v_{2n+1}, the label of the vertex $v_3=$ the label of the vertex v_{n+1}, the label of the vertex $v_5=$ the label of the vertex v_{n+2}, ..., the label of the vertex $v_{n-1}=$ the label of the vertex $v_{\frac{3n}{2}-1}$, the label of the vertex $v_2=$ the label of the vertex $v_{\frac{3n}{2}+1}$, the label of the vertex $v_4=$ the label of the vertex $v_{\frac{3n}{2}+2}$, the label of the vertex $v_6=$ the label of the vertex $v_{\frac{3n}{2}+3}$, ..., the label of the vertex $v_{\frac{3n}{2}}=$ the label of the vertex v_{2n} (See Figure 4.20). That is, there exists at least one such closed walk $\overrightarrow{W}$ in $\overleftrightarrow{K}_k$ satisfying the above condition.
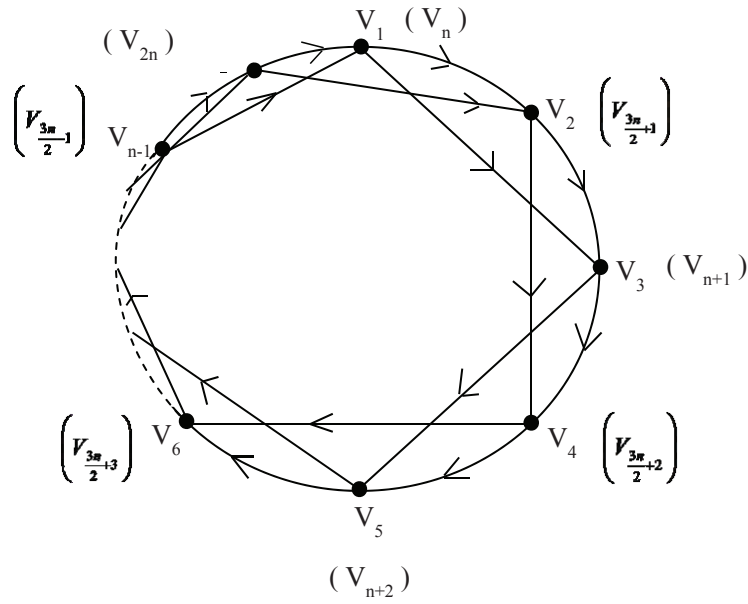
Figure 4.20: Circulant digraph $\overrightarrow{G_n}(1,2)$.

When k is odd, for $n = \frac{k^2-k+2}{2}$, we cannot color the vertices of $\overrightarrow{G_n}(1,2)$ with k colors as all the ordered pairs of colors cannot be assigned to the vertices of $\overrightarrow{G_n}(1,2)$ to obtain a proper complete coloring. (Since either all the ordered pairs of colors cannot be assigned to colors the vertices of $\overrightarrow{G_n}(1,2)$ or the ordered pair (i,i) of one particular color appears on at least one arc of $\overrightarrow{G_n}(1,2)$.)

As k colors are used to color the vertices of $\overrightarrow{G_n}(1,2)$, there are $k(k-1)$ ordered pairs of colors. In $\overrightarrow{G_n}(1,2)$, at each vertex there will be two incoming arcs and two outgoing arcs. Hence there should be even number of ordered pairs having one particular color. There are $k-1$ ordered pairs associated with each color. When k is even, $k-1$ will be odd and hence in order to use all the ordered pairs of one particular color, we have to repeat some ordered pairs. Hence for k colors, we must repeat k ordered pairs. Thus, when k is even, for a circulant digraph with less than $\frac{k^2}{2}$ vertices, we require $k-1$ colors. Hence the proof.

$\square$

Figures 4.21 and 4.22 are illustrative examples of the above result.
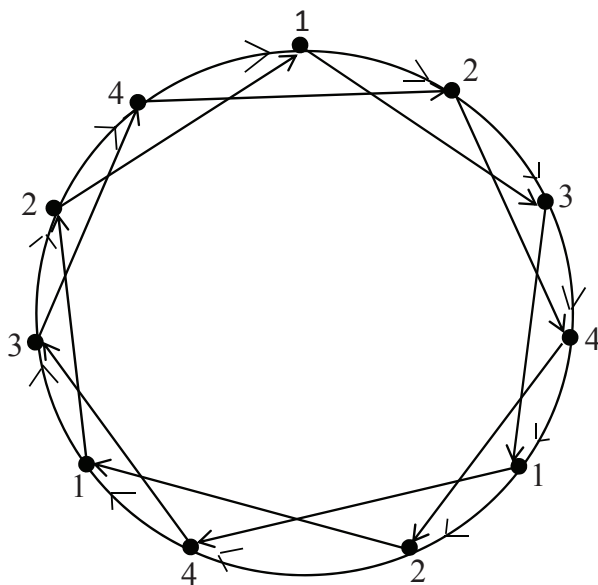
Figure 4.21: A proper complete coloring of $\overrightarrow{G}_{11}(1, 2)$.
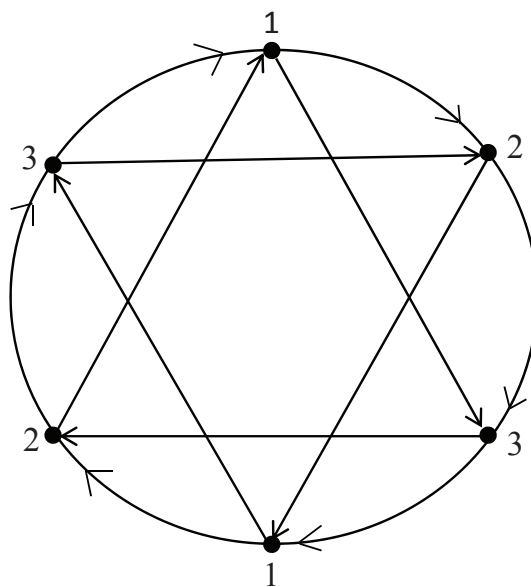
Figure 4.22: A proper complete coloring of $\overrightarrow{G}_{6}(1, 2)$.

Chapter 5

Set Colorings of Digraphs

The notion of set coloring of a graph was introduced in 2009 by Hegde (2009). In its original version, both vertices and edges of an undirected graph are colored with finite sets of positive integers. The color of an edge (u, v) is given by the symmetric difference of the colors of u and v. A graph is said to be **set colorable** if there exists an assignment of colors on the vertices such that both conditions are fulfilled:

(i) all the colors on the vertices are distinct

(ii) all the colors on the edges are distinct.

In this chapter we focus on a type of vertex coloring called set colorings of digraphs. We give some necessary conditions for a digraph to admit a strong set coloring (proper set coloring). We characterize strongly (properly) set colorable digraphs such as directed stars, directed bistars etc. Also, we find the construction of strongly (properly) set colorable directed caterpillars.

5.1 Introduction

We have extended the idea of set coloring to directed graphs as follows:

Let X be a nonempty set of colors, 2^X denote the set of all possible combinations of colors (or power set) of X and $Y(X) = 2^X \setminus \emptyset$.

Let $D = (V, E)$ be a digraph with n vertices and m arcs and let X be a nonempty set of colors. We define a function f on the vertex set V of D as an assignment of subsets of X to the vertices of D and given such a function f on the vertex set V, we define f^* on the set of arcs E as an assignment of the colors $f^*(e) = f(v) - f(u)$ to the arc $e = (u, v)$ of D.

Definition 5.1.1. *Let $f(D) = \{f(u) : u \in V\}$ and $f^*(D) = \{f^*(e) : e \in E\}$. We call f a **set coloring** of D if both $f(D)$ and $f^*(D)$ are injective functions. A digraph is called **set colorable** if it admits a set coloring.*

Definition 5.1.2. *A set coloring f of D is called a **strong set coloring** if $f(D) \cap f^*(D) = \emptyset$ and $f(D) \cup f^*(D) = Y(X)$. If D admits such a coloring then D is called a **strongly set colorable** digraph.*

Definition 5.1.3. *A set coloring f is called a **proper set coloring** if $f^*(D) = Y(X)$. If a digraph D admits such a set coloring, then it is called a **proper set colorable** digraph.*

The set coloring number $\sigma(D)$ of a digraph D is the least cardinality of a set X with respect to which D has a set coloring. Further if $f : V \to 2^X$ is a set coloring of D with $|X| = \sigma(D)$, we call f an **optimal set coloring** of D.

Figure 5.1 gives examples of (a) strongly, (b) properly, (c) non-strongly and non-properly set colorable digraphs.

Theorem 5.1.4. *For any digraph D, $\lceil log_2(m + 1) \rceil \leq \sigma(D) \leq n - 1$, where m and n are the number of arcs and vertices of D respectively and $\lceil x \rceil$ denotes the least integer not less than the real number x and the bounds are best possible.*
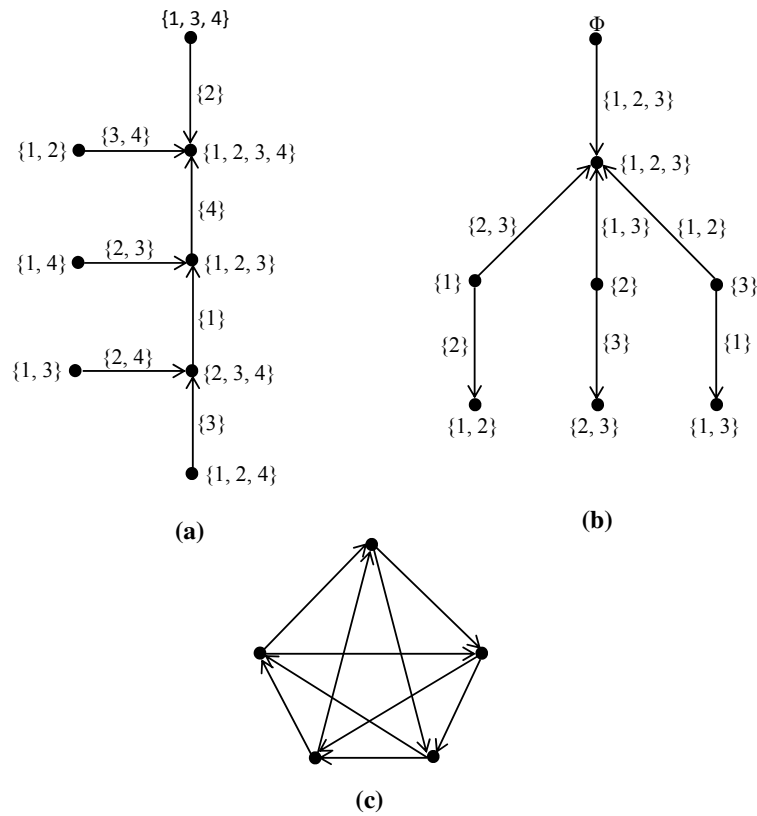
Figure 5.1: Stongly, Properly, Non-strongly and non-properly set colored digraphs

In the next section, we give some necessary conditions for the strong and proper set colorings of digraphs

5.2 Necessary conditions for strong (proper) set colorings of digraphs

Since all the nonempty subsets have to appear in any strong set coloring of a (n, m)-digraph D, a necessary condition for D to be strongly set colorable is that $n + m + 1 = 2^k$, for the positive integer $|X| = k$. This necessary condition immediately yields that any oriented cycle is not strongly set colorable. Also, we observe that the above

condition is not sufficient for saying that a digraph D is strongly set colorable as a unipath of length 7 satisfies the condition for $k = 4$, but one can verify that it is not strongly set colorable.

Similarly, a necessary condition for D to be properly set colorable is that $m + 1 = 2^k$, where m is the number of arcs and $|X| = k$. From the necessary condition it follows that any oriented cycles of lengths not equal to $2^k - 1$ are not properly set colorable.

Let $D(V, E)$ be any digraph and let K_1 be the complete graph with one vertex say v. Then we define the digraph $D + K_1$ as $D \cup K_1$ together with all the arcs joining from the vertex v to the vertices of V.

The following result gives a natural link between strongly set colorable and properly set colorable digraphs.

Theorem 5.2.1. *A digraph D is strongly set colorable if and only if $D + K_1$ has a proper set coloring F such that $F(v) = \phi$.*

Proof. Let f be a strong set coloring of D. Then, define the restricted map F by

$$F(u) = \begin{cases} f(u) & if\ u \in V, \\ \phi & if\ u = v. \end{cases}$$

Since f is a strong set coloring of D, the arcs of $D + K_1$ having the form (v, u), where $u \in V(D)$ will receive $f(u)$. So F turns out to be a required proper set coloring of $D + K_1$.

Conversely, if $D_1 = D + K_1$ has a proper set coloring F with $F(v) = \phi$, then the removal of v from D_1 obviously results in a strong set coloring of D.

$\square$

The following results give stronger necessary conditions for a strong (proper) set coloring of digraphs.

Theorem 5.2.2. *Every strongly set colorable digraph D has a sink.*

Proof. Let D be a digraph with a strong set coloring f with respect to a set X having k colors. Since D is strongly set colorable, either the full set X is obtained on the arc or assigned to any vertex of D. As the empty set cannot be assigned to any vertex of D, one can observe that the full set X cannot be obtained on any arc of D. Hence the set X has to be assigned to a vertex, say v of D. Then we have two cases, namely, $od(v) \geq 1$ or $od(v) = 0$. Suppose $od(v) \geq 1$, that is if u be any vertex of D such that (v, u) is an arc in D, then $f^*((v, u)) = f(u) - f(v) = f(u) - X = \phi$, a contradiction as D is strongly set colorable. Therefore, $od(v) = 0$. As the set X has to be assigned to a vertex only, v has to be a sink. Hence every strongly set colorable digraph D has a sink.

$\square$

Corollary 5.2.3. *Symmetric digraphs, complete symmetric digraphs and conservative digraphs are not strongly set colorable.*

Theorem 5.2.4. *Every properly set colorable digraph has a source and a sink.*

Proof. Let the digraph D have a proper set coloring f with respect to a set X of cardinality k. Let $\{v_1, v_2, ..., v_n\}$ be the vertices of D such that $f(v_i) = A_i$, $1 \leq i \leq n$ and $A_i \in Y(X) \cup \phi$. As D is properly set colorable, we have $f^*(D) = \{A_j - A_i : (v_i, v_j) \in E\} = Y(X)$. Let A be the full set of X and B be the empty set of X. By the definition of f^*, A can appear on the arc (u, v), where $u, v \in V(D)$ only if $f(v) = A$ and $f(u) = B$. Suppose assume that $id(u) \geq 1$. That is, if u is any vertex of D such that for any vertex w in D, (w, u) is an arc in D, then $f^*(e) = f(u) - f(w) = B - f(w) = \phi$ which is not possible. Hence $id(u) = 0$. Hence B can be assigned to a vertex only if it is a source. Also, assume that $od(v) \geq 1$. That is, if v is any vertex of D such that for any vertex y in D, (v, y) is an arc in D, then $f^*(e) = f(y) - f(v) = f(y) - A = \phi$ which is not possible. Hence $od(v) = 0$.

Hence A can be assigned to a vertex only if it is a sink. Hence every properly set colorable digraph has a source and a sink.

$\square$

In the next section, we characterize the strong (proper) coloring of some familiar classes of digraphs.

5.3 Characterization of strong (proper) coloring of some classes of digraphs

We start with the following result for a unipath on n vertices.

Theorem 5.3.1. *No unipath of length greater than or equal to 2 is properly set colorable.*

Proof. Let $\overrightarrow{P_n}$ be a unipath with n vertices. For the unipath $\overrightarrow{P_n}$ to be properly set colorable, all the non empty subsets of X should appear on the arcs. It is possible only when the full set is assigned to the sink and the empty set is assigned to the source. But for a unipath of length greater than or equal to 2, it is not possible as all the non empty subsets will not appear on the arcs. Hence it is not properly set colorable.

$\square$

Figure 5.2 is an illustration of a semipath of length 3 which is properly set colorable.
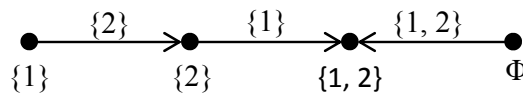
Figure 5.2: Proper set coloring of a directed path $\overrightarrow{P}_3$.

Theorem 5.3.2. *Let $\overrightarrow{S_n}$ be a directed star with n vertices in which either $id(v) = 0$ or $od(v) = 0$, where v is the central vertex. Then $\overrightarrow{S_n}$ is strongly set colorable if and only if $n = 2^{k-1}$.*

Proof. Let $\overrightarrow{S_n}$ be a directed star with n vertices in which either $id(v) = 0$ or $od(v) = 0$, where v is the central vertex. Let $\overrightarrow{S_n}$ be a digraph with strong set coloring f with respect to a set X having k colors. Then it follows that $|V(\overrightarrow{S_n})| + |E(\overrightarrow{S_n})| = 2^k - 1$.
i.e., $n + n - 1 = 2^k - 1$.
i.e., $n = 2^{k-1}$.
Conversely, let $\overrightarrow{S_n}$ be a directed star with n vertices, where $n = 2^{k-1}$.

Case (i) Let $id(v) = 0$, where v is the central vertex of $\overrightarrow{S_n}$. Let $X = \{1, 2, ..., k\}$ and $S = \{a\}$, $a \in X$. Assign the set S to the central vertex and the subsets of X containing the element a to the remaining vertices of $\overrightarrow{S_n}$ in a one-to-one manner. Then it can be easily verified that the assignment is a strong set coloring of $\overrightarrow{S_n}$.

Case (ii) Let $od(v) = 0$, where v is the central vertex of $\overrightarrow{S_n}$. Let $X = \{1, 2, ..., k\}$ and assign the set X to the central vertex. One can observe that, when the set difference is applied from the set X to $(k - 2)$ - element subsets, we obtain 2 - element subsets, $(k - 3)$ - element subsets, we obtain 3 - element subsets,..., $(\frac{(k-2)}{2})$ - element subsets, we obtain $(\frac{(k+2)}{2})$ - element subsets when k is even and $(\frac{(k-1)}{2})$ - element subsets, we obtain $(\frac{(k+1)}{2})$ - element subsets when k is odd.

Case (a) Let k be odd. Then assign $(k - 1)$ - element subsets, $(k - 2)$ - element subsets, $(k-3)$ - element subsets,..., $(\frac{(k-1)}{2})$ - element subsets to the remaining $2^{k-1} - 1$ vertices. Then it is easy to verify that the assignment is a strong set coloring of $\overrightarrow{S_n}$ when k is odd.

Case (b) Let k be even. Then assign $(k - 1)$ - element subsets, $(k - 2)$ - element subsets, $(k - 3)$ - element subsets,..., half of $(\frac{k}{2})$ - element subsets to the remaining $2^{k-1} - 1$ vertices of $\overrightarrow{S_n}$. Then it is easy to verify that the assignment is a strong set coloring of $\overrightarrow{S_n}$ when k is even. $\qquad\square$

Figure 5.3 displays the strong set coloring of a directed star $\overrightarrow{S_8}$ when (a) $id(v) = 0$; (b) $od(v) = 0$, where v is the central vertex.
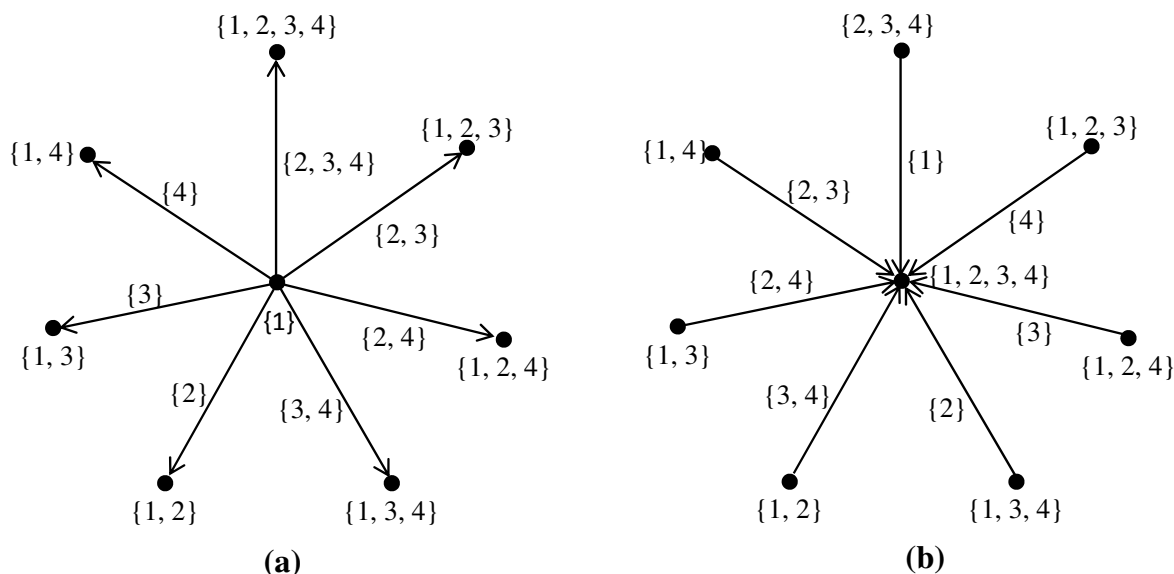
Figure 5.3: Strong set coloring of $\overrightarrow{S_8}$.

Theorem 5.3.3. *Let $\overrightarrow{S_n}$ be a directed star with the central vertex v such that either $id(v) = 0$ or $od(v) = 0$. Then $\overrightarrow{S_n}$ is properly set colorable if and only if $n = 2^k$.*

Proof. Let $\overrightarrow{S_n}$ be a directed star with n vertices and $(n-1)$ arcs. Let v be the central vertex of $\overrightarrow{S_n}$. Let $\overrightarrow{S_n}$ be be a properly set colorable digraph with respect to a set X having k colors. Then $|E(\overrightarrow{S_n})| = 2^k - 1$. This implies

$n - 1 = 2^k - 1$.

i.e., $n = 2^k$.

Conversely, let $\overrightarrow{S_n}$ be a directed star with n vertices such that $n = 2^k$. Let v be the central vertex of $\overrightarrow{S_n}$.

Case (i) Let $id(v) = 0$. Let $X = \{1, 2, ..., k\}$. Assign the empty set to the central vertex v and the remaining subsets of X to the remaining vertices of $\overrightarrow{S_n}$. Then one can observe that the non empty subsets of X will appear on the arcs. Hence $\overrightarrow{S_n}$ is

properly set colorable.

Case (ii) Let $od(v) = 0$. Let $X = \{1, 2, ..., k\}$ and assign the set X to the central vertex and the remaining subsets of X to the remaining vertices of $\overrightarrow{S_n}$. Then one can observe that the nonempty subsets of X will appear on the arcs. Hence $\overrightarrow{S_n}$ is properly set colorable.

$\square$

Figure 5.4 displays the proper set coloring of a directed star $\overrightarrow{S_8}$ when (a) $id(v) = 0$; (b) $od(v) = 0$, where v is the central vertex.
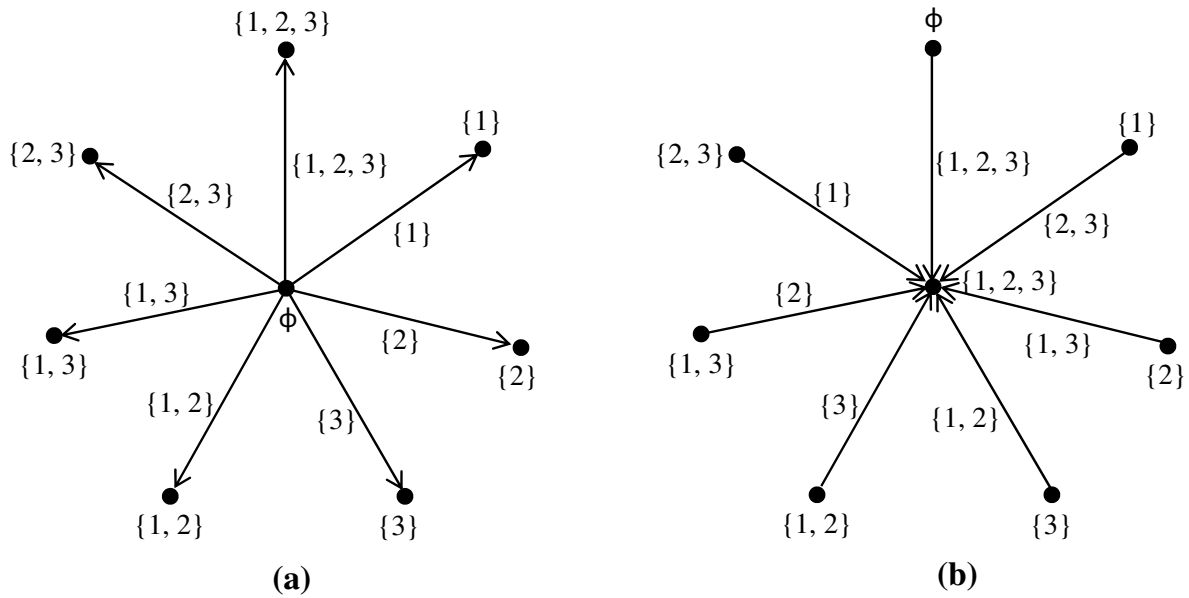
Figure 5.4: Proper set coloring of $\overrightarrow{S_8}$.

Definition 5.3.4. *Let $\overrightarrow{K}_{1,n}$ be a directed star such that $od(v) = 0$, where v is the central vertex. Then $\overrightarrow{B}_{n,n}$ is a digraph obtained from two copies of $\overrightarrow{K}_{1,n}$ by joining the vertices of maximum degree by an arc, which is called a **directed bistar**.*

Next, we characterize the strong (proper) set coloring of directed bistar.

Theorem 5.3.5. *A directed bistar $\overrightarrow{B}_{n,n}$ is strongly set colorable if and only if $n = 2^{k-2} - 1$.*

Proof. A directed bistar $\overrightarrow{B}_{n,n}$ has $2(n+1)$ vertices and $2(n+1)-1$ arcs. Let $\overrightarrow{B}_{n,n}$ be a strongly set colorable digraph with respect to a set X having k colors. Then

$|V(\overrightarrow{B}_{n,n})| + |E(\overrightarrow{B}_{n,n})| = 2^k - 1$

$\implies 2(n+1) + 2(n+1) - 1 = 2^k - 1$

$\implies n = 2^{k-2} - 1.$

Conversely, let $\overrightarrow{B}_{n,n}$ be a directed bistar such that $n = 2^{k-2} - 1$. Let $X = \{1, 2, ..., k\}$. Also, let $X_1 = \{1, 2, ..., k\}$, the full set of X and $X_2 = $ a subset containing $k - 1$ elements of X which doesn't contain the element a, $a \in X$. When we apply the set difference from the set X to $(k - 2)$ - element subsets, we get 2 - element subsets, $(k - 3)$ - element subsets, we get 3 - element subsets, ..., $(\frac{k-2}{2})$ - element subsets, we get $(\frac{k+2}{2})$ - element subsets when k is even and $(\frac{k-1}{2})$ - element subsets, we get $(\frac{k+1}{2})$ - element subsets when k is odd. Assign the set X_1 to the sink of $\overrightarrow{B}_{n,n}$, that is a vertex say, v of $\overrightarrow{B}_{n,n}$ when $od(v) = 0$. Also, assign the set X_2 to the vertex say u which is adjacent to v and $id(u) = n$ and $od(u) = 1$. Assign all the subsets of X which contains the element a, except the singleton set a, to the remaining vertices of $\overrightarrow{B}_{n,n}$. Then one can observe that the elements on the arcs are the subsets of set X. Hence $\overrightarrow{B}_{n,n}$ is strongly set colorable.

$\square$

Figure 5.5 is an illustration of the above result.

Theorem 5.3.6. *A directed bistar $\overrightarrow{B}_{n,n}$ is properly set colorable if and only if $n = 2^{k-1} - 1$.*

Proof. A directed bistar $\overrightarrow{B}_{n,n}$ has $2(n + 1)$ vertices and $2(n + 1) - 1$ arcs. Let $\overrightarrow{B}_{n,n}$ be properly set colorable digraph with respect to a set X having k colors. Then $|E(\overrightarrow{B}_{n,n})| = 2^k - 1$. This implies
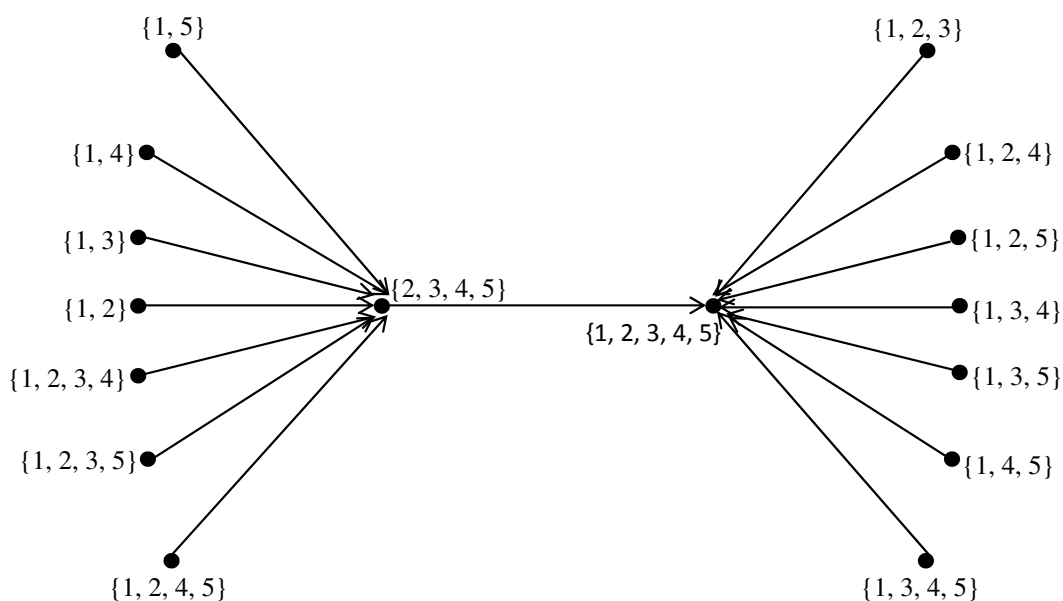
Figure 5.5: Strong set coloring of $\overrightarrow{B}_{7,7}$.

$2(n+1) - 1 = 2^k - 1$.

i.e., $\implies n = 2^{k-1} - 1$.

Conversely, let $\overrightarrow{B}_{n,n}$ be a directed bistar such that $n = 2^{k-1} - 1$. Let $X = \{1, 2, ..., k\}$. Also, let $X_1 = \{1, 2, ..., k\}$, the full set of X and $X_2 = $ a subset containing $k - 1$ elements of X which doesn't contain the element a, $a \in X$. Then assign the set X_1 to the sink of $\overrightarrow{B}_{n,n}$, that is a vertex say, v of $\overrightarrow{B}_{n,n}$ where $od(v) = 0$. Also, assign the set X_2 to the vertex say u which is adjacent to v and $id(u) = n$ and $od(u) = 1$. Assign all the subsets of X which contains the element a to the sources of the vertex u and the other subsets (the subsets of X that doesn't contain the element a) to the sources of the vertex v. Then one can observe that the elements on the arcs are the subsets of set X. Hence $\overrightarrow{B}_{n,n}$ is properly set colorable.

$\square$

Figure 5.6 is an illustration of the above result.
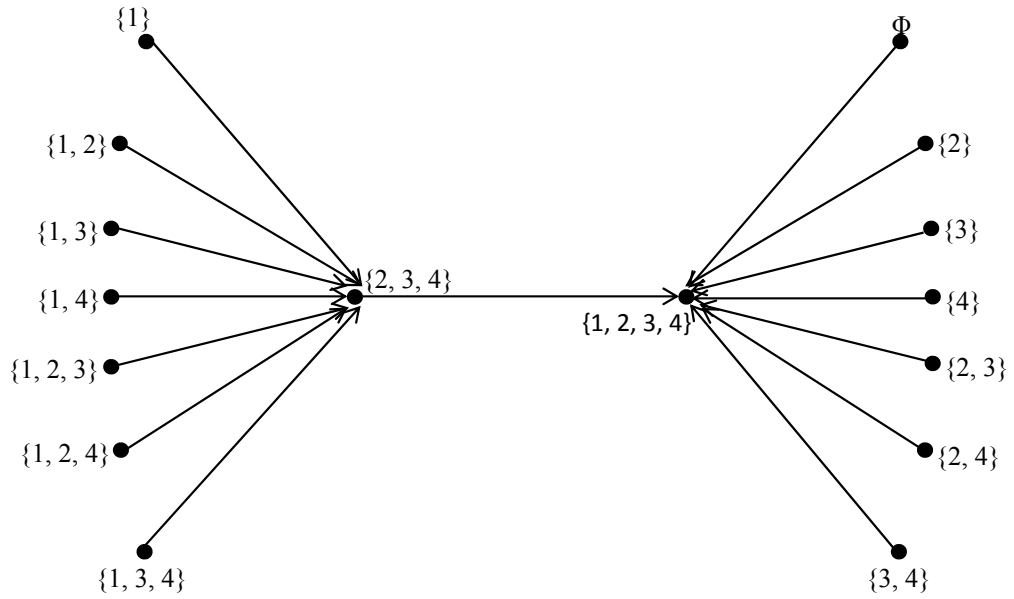
Figure 5.6: Proper set coloring of $\overrightarrow{B}_{7,7}$.

Theorem 5.3.7. *Let $\overrightarrow{T_n^l}$ be a directed complete n-ary tree with the orientation either from the vertices of level l to the vertices of level $l+1$ or vice versa and l denotes the level of the tree , $l = 1, 2, ...$ Then $\overrightarrow{T_n^l}$ is strongly set colorable if and only if $n = 2^\alpha - 1$ and $l = 2$, where α is any positive integer.*

Proof. We know that $\overrightarrow{T_n^l}$ has $\frac{n^l-1}{n-1}$ vertices and $\frac{n(n^{l-1}-1)}{n-1}$ arcs. Suppose that $D = \overrightarrow{T_n^l}$ is strongly set colorable with respect to a set X of cardinality k. Then we obtain $|V(D)| + |E(D)| = 2^k - 1$. This implies that

$$\left(\frac{n^l-1}{n-1}\right) + \left(\frac{n(n^{l-1}-1)}{n-1}\right) = 2^k - 1.$$

i.e.,

$$\frac{2n^l - n - 1}{n-1} = 2^k - 1.$$

i.e.,

$$\frac{n^l - 1}{n - 1} = 2^{k-1}. \tag{5.3.1}$$

i.e.,

$$n^l = 2^{k-1}(n - 1) + 1. \tag{5.3.2}$$

Since $2^{k-1}(n-1)$ is always even, the right hand side of (5.3.2) is odd. Hence left hand side of (5.3.2) also should be odd. But, when n is even, n^l is even and hence (5.3.2) doesn't hold. Hence n should be even. Therefore, equation(5.3.1) holds only if n is odd. Thus, no complete n-ary tree D is strongly set colorable when n is even. Also, from (5.3.1), it follows that n is odd and hence l is even. Thus, from $\frac{n^l-1}{n-1} = 2^{k-1}$, we obtain $(1 + n + n^2 + ... + n^{l-1}) = 2^{k-1}$ or

$$(1 + n)(1 + n^2 + n^4 + ... + n^{l-2}) = 2^{k-1} \tag{5.3.3}$$

which implies that $(1 + n) = 2^\alpha$, α is a positive integer. Thus from equation(5.3.3), we obtain

$$(1 + n^2 + n^4 + ... + n^{l-2}) = 2^{k-\alpha-1}. \tag{5.3.4}$$

One can write equation(5.3.4) as $(1 + n^2)(1 + n^4 + n^8 + ... + n^{l-4}) = 2^{k-\alpha-1}$ which implies that $1 + n^2 = 2^\beta$. Substituting the value of n from $(1 + n) = 2^\alpha$, we obtain $1 + (2^\alpha - 1)^2 = 2^\beta$.

i.e., $2^{2\alpha} - 2^{\alpha+1} + 2 = 2^\beta$.

i.e., $2^{2\alpha-1} - 2^\alpha + 1 = 2^{\beta-1}$.

i.e., $2^{2\alpha-1} - 2^{\beta-1} = 2^\alpha - 1$ which implies that $2^\alpha - 1$ is even or $\alpha = 0$ or $n = 0$, a contradiction. Thus, $1 + n = 2^\alpha$ or $n = 2^\alpha - 1$. Also, from equation(5.3.4) we obtain $1 + 2^{(k-\alpha-1)}$ or $m = \alpha + 1$ and also $l = 2$.

Conversely, suppose that $n = 2^\alpha - 1$ and $l = 2$. Then D reduces to a star $K_{1,2^\alpha - 1}$. The proof follows from Theorem 5.3.2.

$\square$

The following theorem can be proved using the analogous arguments as in Theorem 5.3.7.

Theorem 5.3.8. *Let $\overrightarrow{T_n^l}$ be a directed complete n-ary tree as defined in the previous theorem. Then $\overrightarrow{T_n^l}$ is properly set colorable if and only if $n = 2^\alpha - 1$ and $l = 2$, where α is any positive integer.*

In the next section, we give an embedding of a unicycle $\overrightarrow{C_n}$ with n vertices as an induced subgraph of a strongly (properly) set colorable digraph.

5.4 Embedding of a unicycle as strongly (properly) set colorable digraph

Generally, the digraphs that cannot be strongly (properly) set colorable, can be embedded as an induced subgraph of a strongly (properly) set colorable digraphs. One can see that symmetric digraphs and complete symmetric digraphs cannot be embedded as an induced subgraph of strongly (properly) set colorable digraphs.

In this section, we give an embedding of a unicycle as an induced subgraph of a strongly (properly) set colorable digraph, thereby showing that strongly (properly) set colorable digraphs cannot be characterized using forbidden subgraphs.

First, we give an embedding of a unicycle as an induced subgraph of a strongly set colorable digraph.

Let $\overrightarrow{C_n}$ be a unicycle with n vertices. Let $\overrightarrow{K}_{1,2^{n-1}-n-1}$ be a directed star with $(2^{n-1} - n)$ vertices and $od(v) = 0$, where v is the central vertex. Since a unicycle is

not strongly set colorable, consider the union of a unicycle $\overrightarrow{C_n}$ and the directed star $\overrightarrow{K}_{1,2^{n-1}-n-1}$ as $D = \overrightarrow{C_n} \cup \overrightarrow{K}_{1,2^{n-1}-n-1}$. Then D can be strongly set colorable and is disconnected and has two components. Let X be the set of n colors. Assign the $(n-1)$ - element subsets to the vertices of $\overrightarrow{C_n}$, then the 1 - element (singleton) subsets will be assigned on the arcs of $\overrightarrow{C_n}$. Also, assign the set X to the central vertex of $\overrightarrow{K}_{1,2^{n-1}-n-1}$ and assign $(n-2)$ - element subsets, $(n-3)$ - element subsets,..., $(\frac{n-2}{2})$ - element subsets and half of $(\frac{n}{2})$ - element subsets when n is even and $\lfloor \frac{n}{2} \rfloor$-element subsets when n is odd and $n \neq 5$ to the vertices of $\overrightarrow{K}_{1,2^{n-1}-n-1}$. Also, when $n = 5$, assign $(n-2)$-element subsets to the vertices of $\overrightarrow{K}_{1,2^{n-1}-n-1}$. The remaining subsets of X will appear on the arcs of $\overrightarrow{K}_{1,2^{n-1}-n-1}$. Then one can observe that D is strongly set colorable.

An illustrative example for the embedding of a unicycle $\overrightarrow{C_5}$ as an induced subgraph of a strongly set colorable digraph is given in Figure 5.7.
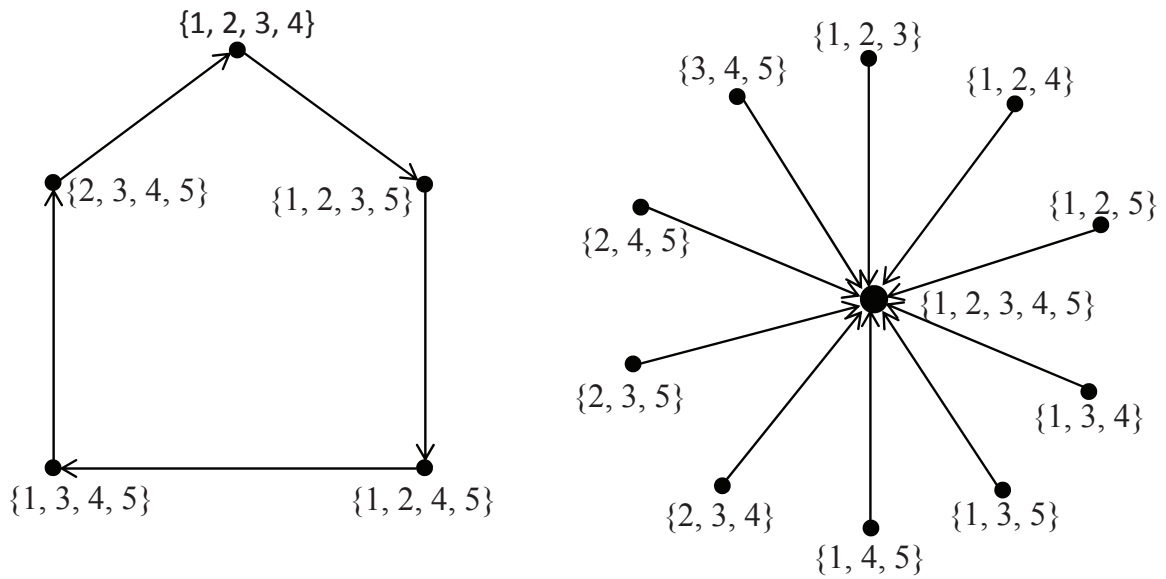

Figure 5.7: Strong set coloring of embedded unicycle $\overrightarrow{C}_5$.

Next, we give an embedding of a unicycle as an induced subgraph of a properly set colorable digraph.

Let $\overrightarrow{C_n}$ be a unicycle with n vertices. Let $\overrightarrow{K}_{1,2^n-n-1}$ be a directed star with (2^n-n) vertices and $od(v) = 0$, where v is the central vertex. Since a unicycle is not properly set colorable, consider the union of a unicycle $\overrightarrow{C_n}$ and the directed star $\overrightarrow{K}_{1,2^n-n-1}$ as $D = \overrightarrow{C_n} \cup \overrightarrow{K}_{1,2^n-n-1}$. Then D can be properly set colorable and is disconnected and has two components. Let X be the set of n colors. Assign the $(n-1)$ - element subsets to the vertices of $\overrightarrow{C_n}$, then the 1 - element (singleton) subsets will be assigned on the arcs of $\overrightarrow{C_n}$. Also, assign the set X to the central vertex of $\overrightarrow{K}_{1,2^n-n-1}$ and assign all the remaining subsets of X (except $(n-1)$ - element subsets) to the vertices of $\overrightarrow{K}_{1,2^n-n-1}$. Then one can observe that D is properly set colorable.

An illustrative example for the embedding of a unicycle $\overrightarrow{C_4}$ as an induced subgraph of a properly set colorable digraph is given in Figure 5.8.
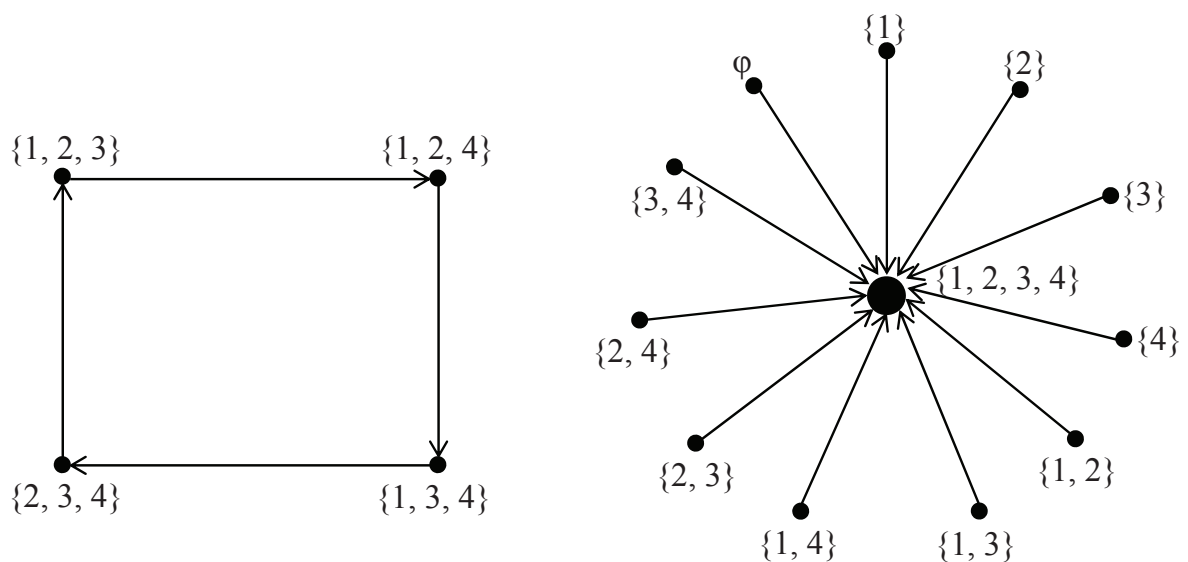

Figure 5.8: Proper set coloring of embedded unicycle $\overrightarrow{C}_4$.

In the next section, we give a construction of a bigger properly set colored digraph from a properly set colored directed tree. Also, we give a construction of strongly (properly) set colorable directed caterpillars.

5.5 Construction of strongly (properly) set colorable directed caterpillars

Given below is a construction of a bigger properly set colored digraph from a properly set colored directed tree.

Let $\overrightarrow{T}(V, E)$ be a properly set colorable directed tree with respect to a set X of cardinality k. Let K_1 be the complete graph with one vertex say v. Then we define the digraph $\overrightarrow{T} + K_1$ as $\overrightarrow{T} \cup K_1$ together with all the arcs joining from the vertices of V to the vertex v. Let X' be the set containing $k + 1$ elements. Now, assign the set X' to the vertex v. As $\overrightarrow{T}$ is properly set colorable, all the subsets of X appear on the vertices and all the nonempty subsets of X appear on the arcs of $\overrightarrow{T}$. Now, in $\overrightarrow{T} + K_1$, one can observe that all the nonempty subsets of X' appear on the arcs of $\overrightarrow{T} + K_1$. Hence $\overrightarrow{T} + K_1$ is properly set colorable.

Next, we give the construction of strongly set colorable directed caterpillars.

Definition 5.5.1. *A **caterpillar** is a tree which gives a path when all its pendant vertices are deleted. A **directed caterpillar** is a oriented tree which gives a unipath when all its pendant vertices (i.e. when either indegree $= 1$ or outdegree $= 1$) are deleted.*

Next, we give the construction of an infinite family of strongly set colorable directed caterpillars are given below.

Let X_1 be a nonempty set with $|X_1| = m_1$, where $m_1 \geq 3$ is a positive integer. Consider the directed star $K_{1,2^{m_1-1}-1} = T_0(m_1)$ (say) with $id(v) = 0$, where v is the central vertex. Let $v_1, v_2, \cdots, v_{2^{m_1-1}-1}$ be the pendant vertices of $T_0(m_1)$. We define a mapping $f_1 : V(T_0(m_1)) \to 2^{X_1}$ as follows:

$f_1(v) = x_0$, where $x_0 \in X_1$

$f_1(v_i) = A_r$, where A_r is a subset of X_1 such that the element $x_0 \in A_r$ for $i < 2^{m_1-1}-1$.

$f_1(v_{2^{m_1-1}-1}) = X_1$.

Clearly, f_1 and f_1^* are injective functions. Let X_2 be a set of cardinality m_2, where $m_2 > m_1$. Introduce new vertices, say $u_{1,1}, u_{1,2}, \cdots, u_{1,k_1}$, where $k_1 = 2^{m_2-1} - 2^{m_1-1}$ and join each of them to $v_{2^{m_1-1}-1}$ such that $od(v_{2^{m_1-1}-1}) = 0$. Let the resulting directed caterpillar be denoted by $T_1(m_2)$ and define the mapping $f_2 : V(T_1(m_2)) \to 2^{X_2}$ as follows:

$f_2(v) = x_0 \cup m_2 = A$

$f_2(v_i) = A_r \cup m_2 = A'_r$ for $i < 2^{m_1-1} - 1$.

$f_2(v_{2^{m_1-1}-1}) = X_1 \cup m_2 = X_2$.

$f_2(u_{1,i'}) = B_r$, where B_r is a subset of X_2 other than A'_r containing the element m_2 for $i' < k_1$.

$f_2(u_{1,k_1}) = X_2 - x_0 = B$.

Let $f_2^* : E(T_1(m_2)) \to 2^{X_2}$ denote the induced edge function defined by $f_2^*((u, v)) = f_2(v) - f_2(u)$, where $(u, v) \in E(T_1(m_2))$. Then one can easily verify that both f_2 and f_2^* are injective functions and hence $T_1(m_2)$ is strongly set colorable.

Let X_3 be a set of cardinality m_3, where $X_1 \subset X_2 \subset X_3$ and $m_1 < m_2 < m_3$. Now, change the direction of the arc $(u_{1,k_1}, v_{2^{m_1-1}-1})$ to $(v_{2^{m_1-1}-1}, u_{1,k_1})$. Introduce $2^{m_3-1} - 2^{m_2-1}$ new vertices, say $u_{2,1}, u_{2,2}, \cdots, u_{2,k_2}$, where $k_2 = 2^{m_3-1} - 2^{m_2-1}$ and join each of them to u_{1,k_1} such that $od(u_{1,k_1}) = 0$. Let the resulting directed caterpillar be denoted by $T_2(m_3)$. Define the mapping $f_3 : V(T_2(m_3)) \to 2^{X_3}$ as follows:

$f_3(v) = A \cup m_3 = A'$.

$f_3(v_i) = A'_r \cup m_3 = A''_r$ for $i < 2^{m_1-1} - 1$.

$f_3(v_{2^{m_1-1}-1}) = X_2$.

$f_3(u_{1,i'}) = B_r \cup m_3 = B'_r$ for $i' < k_1$.

$f_3(u_{1,k_1}) = B \cup x_0, m_3 = X_3$.

$f_3(u_{2,i''}) = C_r$, where C_r is a subset of X_3 other than A''_r and B'_r containing the element m_3 for $i'' < k_2$.

$f_3(u_{2,k_2}) = X_3 - x_0 = C$.

Let $f_3^* : E(T_2(m_3)) \to 2^{X_3}$ denote the induced edge function defined by $f_3^*((u,v)) = f_3(v) - f_3(u)$, where $(u,v) \in E(T_2(m_3))$. Then one can easily verify that $T_2(m_3)$ is strongly set colorable.

Next, change the direction of the arc (u_{2,k_2}, u_{1,k_1}) to (u_{1,k_1}, u_{2,k_2}). Introduce $2^{m_4-1} - 2^{m_3-1}$ new vertices, say $u_{3,1}, u_{3,2}, \cdots, u_{3,k_3}$, where $k_3 = 2^{m_4-1} - 2^{m_3-1}$ and join each of them to u_{2,k_2} such that $od(u_{2,k_2}) = 0$. Let X_4 be a set of cardinality m_4, where $X_1 \subset X_2 \subset X_3 \subset X_4$ and $m_1 < m_2 < m_3 < m_4$. Let the resulting directed caterpillar be denoted by $T_3(m_4)$. Define the mapping $f_4 : V(T_3(m_4)) \to 2^{X_4}$ by

$f_4(v) = A' \cup m_4 = A''$.

$f_4(v_i) = A_r'' \cup m_4 = A_r'''$ for $i < 2^{m_1-1} - 1$.

$f_4(v_{2^{m_1-1}-1}) = X_2 \cup m_4$.

$f_4(u_{1,i'}) = B_r' \cup m_4 = B_r''$ for $i' < k_1$.

$f_4(u_{1,k_1}) = X_3$.

$f_4(u_{2,i''}) = C_r \cup m_4 = C_r'$, for $i'' < k_2$.

$f_4(u_{2,k_2}) = C \cup x_0, m_4 = C'$.

$f_4(u_{3,i'''}) = D_r$, where D_r is a subset of X_4 other than A_r''', B_r'' and C_r' containing the element m_4 for $i''' < k_3$.

$f_4(u_{3,k_3}) = X_4 - x_0 = D$.

Let $f_4^* : E(T_3(m_4)) \to 2^{X_4}$ denote the induced edge function defined by $f_4^*((u,v)) = f_4(v) - f_4(u)$, where $(u,v) \in E(T_3(m_4))$. Then one can easily verify that both f_4 and f_4^* are injective functions and hence $T_3(m_4)$ is strongly set colorable.

We may iterate this procedure indefinitely to obtain the strongly set colorable directed caterpillar at the nth step $n = 1, 2, 3, \cdots$, where $X_1 \subset X_2 \subset X_3 \subset \cdots \subset X_n$ and $m_1 < m_2 < \cdots < m_n$ are chosen quite arbitrarily.

An illustrative example for the construction of strongly set colorable directed caterpillar is given in Figure 5.9.
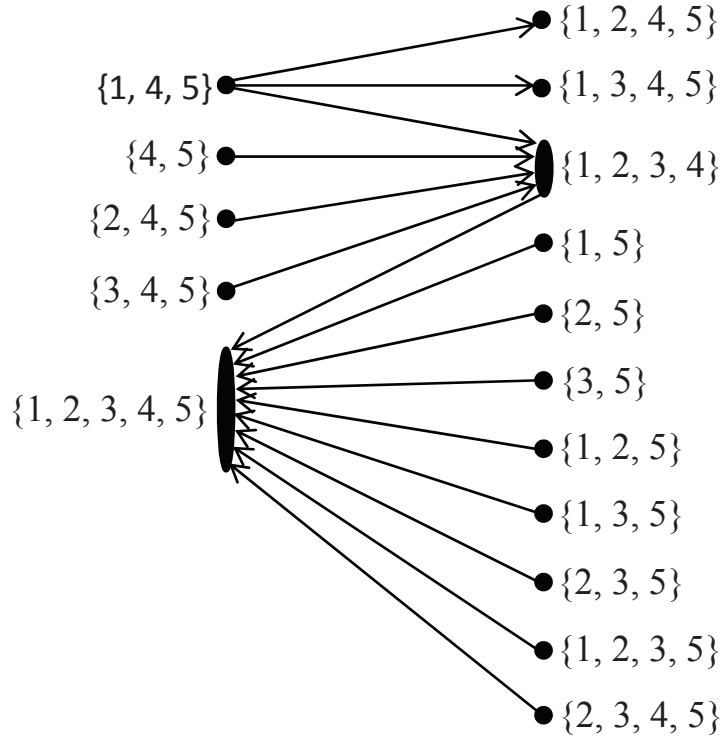
Figure 5.9: Strongly set colorable directed caterpillar.

Construction of an infinite family of properly set colorable directed caterpillars are given below.

Let X_1 be a nonempty set with $|X_1| = m_1$, where $m_1 \geq 2$ is a positive integer. Consider the directed star $K_{1,2^{m_1}-1} = T_0(m_1)$ (say) with $id(v) = 0$, where v is the central vertex. Let $v_1, v_2, \cdots, v_{2^{m_1}-1}$ be the pendant vertices of $T_0(m_1)$. We define a mapping $F_1 : V(T_0(m_1)) \to 2^{X_1}$ as follows:

$F_1(v) = \phi$

$F_1(v_i) = A_r$, where A_r is a nonempty subset of X_1 such that $A_r \neq X_1$ for $i < 2^{m_1} - 1$.

$F_1(v_{2^{m_1}-1}) = X_1$.

Let X_2 be a set of cardinality m_2, where $m_2 > m_1$. Introduce new vertices, say $u_{1,1}, u_{1,2}, \cdots, u_{1,k_1}$, where $k_1 = 2^{m_2} - 2^{m_1}$ and join each of them to $v_{2^{m_1}-1}$ such that $od(v_{2^{m_1}-1}) = 0$. Let the resulting directed caterpillar be denoted by $T_1(m_2)$ and define

the mapping $F_2 : V(T_1(m_2)) \rightarrow 2^{X_2}$ as follows:

$F_2(v) = \phi \cup m_2 = A$

$F_2(v_i) = A_r \cup m_2 = A'_r$ for $i < 2^{m_1} - 1$.

$F_2(v_{2^{m_1}-1}) = X_1 \cup m_2 = X_2$.

$F_2(u_{1,i'}) = B_r$, where B_r is a subset of X_2 other than A'_r without containing the element m_2 and B_r is not a $(m_2 - 1)$-element subset of X_2 for $i' < k_1$.

$F_2(u_{1,k_1}) = X_2 - m_2 = B$.

Let $F_2^* : E(T_1(m_2)) \rightarrow 2^{X_2}$ denote the induced edge function defined by $F_2^*((u,v)) = F_2(v) - F_2(u)$, where $(u,v) \in E(T_1(m_2))$. Then one can easily verify that $F_2^* = Y(X_2)$. Hence $T_1(m_2)$ is properly set colorable.

Let X_3 be a set of cardinality m_3, where $X_1 \subset X_2 \subset X_3$ and $m_1 < m_2 < m_3$. Now, change the direction of the arc $(u_{1,k_1}, v_{2^{m_2}-1})$ to $(v_{2^{m_2}-1}, u_{1,k_1})$. Introduce $2^{m_3} - 2^{m_2}$ new vertices, say $u_{2,1}, u_{2,2}, \cdots, u_{2,k_2}$, where $k_2 = 2^{m_3} - 2^{m_2}$ and join each of them to u_{1,k_1} such that $od(u_{1,k_1}) = 0$. Let the resulting directed caterpillar be denoted by $T_2(m_3)$. Define the mapping $F_3 : V(T_2(m_3)) \rightarrow 2^{X_3}$ as follows:

$F_3(v) = A \cup m_3 = A'$.

$F_3(v_i) = A'_r \cup m_3 = A''_r$ for $i < 2^{m_1} - 1$.

$F_3(v_{2^{m_1}-1}) = X_2$.

$F_3(u_{1,i'}) = B_r \cup m_3 = B'_r$ for $i' < k_1$.

$F_3(u_{1,k_1}) = B \cup m_2, m_3 = X_3$.

$F_3(u_{2,i''}) = C_r$, where C_r is a subset of X_3 other than A''_r and B'_r without containing the element m_3 and C_r is not a $(m_3 - 1)$-element subset of X_3 for $i'' < k_2$.

$F_3(u_{2,k_2}) = X_3 - m_2 = C$.

Let $F_3^* : E(T_2(m_3)) \rightarrow 2^{X_3}$ denote the induced edge function defined by $F_3^*((u,v)) = F_3(v) - F_3(u)$, where $(u,v) \in E(T_2(m_3))$. Then $F_3^* = Y(X_3)$ and hence $T_2(m_3)$ is properly set colorable.

Next, change the direction of the arc (u_{2,k_2}, u_{1,k_1}) to (u_{1,k_1}, u_{2,k_2}). Introduce $2^{m_4} - 2^{m_3}$ new vertices, say $u_{3,1}, u_{3,2}, \cdots, u_{3,k_3}$, where $k_3 = 2^{m_4} - 2^{m_3}$ and join each of them to

u_{2,k_2} such that $od(u_{2,k_2}) = 0$. Let X_4 be a set of cardinality m_4, where $X_1 \subset X_2 \subset X_3 \subset X_4$ and $m_1 < m_2 < m_3 < m_4$. Let the resulting directed caterpillar be denoted by $T_3(m_4)$. Define the mapping $F_4 : V(T_3(m_4)) \to 2^{X_4}$ by

$F_4(v) = A' \cup m_4 = A''$.

$F_4(v_i) = A''_r \cup m_4 = A'''_r$ for $i < 2^{m_1} - 1$.

$F_4(v_{2^{m_1}-1}) = X_2 \cup m_4$.

$F_4(u_{1,i'}) = B'_r \cup m_4 = B''_r$ for $i' < k_1$.

$F_4(u_{1,k_1}) = X_3$.

$F_4(u_{2,i''}) = C_r \cup m_4 = C'_r$, for $i'' < k_2$.

$F_4(u_{2,k_2}) = C \cup m_2, m_4 = X_4$.

$F_4(u_{3,i'''}) = D_r$, where D_r is a subset of X_4 other than A'''_r, B''_r and C'_r without containing the element m_4 and D_r is not a $(m_4 - 1)$-element subset of X_4 for $i''' < k_3$.

$F_4(u_{3,k_3}) = X_4 - m_2 = D$.

Let $F^*_4 : E(T_3(m_4)) \to 2^{X_4}$ denote the induced edge function defined by $F^*_4((u,v)) = F_4(v) - F_4(u)$, where $(u,v) \in E(T_3(m_4))$. Then one can easily verify that $F^*_4 = Y(X_4)$ and hence $T_3(m_4)$ is properly set colorable.

We may iterate this procedure indefinitely to obtain the properly set colorable directed caterpillar at the nth step $n = 1, 2, 3, \cdots$, where $X_1 \subset X_2 \subset X_3 \subset \cdots \subset X_n$ and $m_1 < m_2 < \cdots < m_n$ are chosen quite arbitrarily.

An illustrative example for the construction of properly set colorable directed caterpillar is given in Figure 5.10.
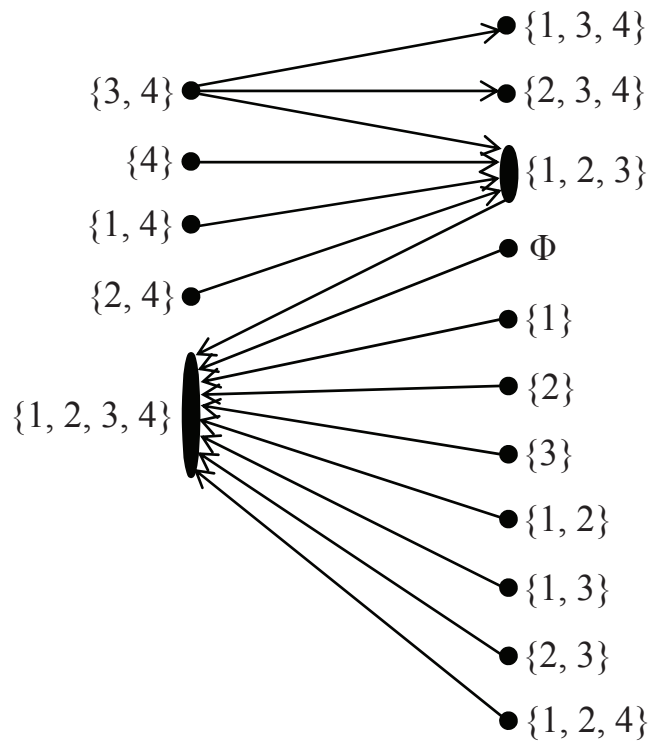
Figure 5.10: Properly set colorable directed caterpillar.

Chapter 6

Conclusion and scope for future research

In this thesis harmonious colorings and complete colorings of graphs have been extended to directed graphs. In addition to some general results, proper harmonious coloring number of several classes of digraphs, namely, oriented cycles, paths, n-ary out-trees, wheels have been discussed. Some lower bound related results have been proved with regard to proper harmonious coloring number of regular digraphs. Similar results have been explored with regard to complete coloring of digraphs. It may be noted that there is a strong relation between harmonious coloring and existence of Eulerian cycles.

Set coloring of digraphs is an interesting extension of coloring wherein colors are sets. Characterization of set coloring of certain classes of digraphs, Embedding of unicycles in a set colored digraph and constructive set coloring of directed caterpillars are some of the findings in the thesis.

The results proved in this thesis are of theoretical interest. Applicability of these types of colorings may be explored as further study. Particularly since the harmonious coloring and complete coloring distinguishes each edge, their applications may be found in network communications.

Bibliography

Acharya, B. D. (1983). "set valuations of a graph and their applications". *MRI Lecture Notes in Applied Mathematics*, 2.

Ali, M. A. and Vivin, J. V. (2007). "harmonious chromatic number of central graph of complete graph families". *Journal of Combinatorics, Information and System Sciences*, 32:221–231.

Asdre, K. and Nikolopoulos, S. D. (2007). "*NP*-completeness results for some problems on subclasses of bipartite and chordal graphs". *Theoretical Computer Science*, 381:248–259.

Balister, P. N., Gyori, E., and Schelp, R. H. (2011). "coloring vertices and edges of a graph by nonempty subsets of a set". *European Journal of Combinatorics*, 32:533–537.

Beivide, R., Herrada, E., Balcazar, J. L., and Arruabarrena, A. (1991). "optimal distance networks of low degree for parallel computers". *IEEE Transactions on Computers*, 40:1109–1124.

Bermond, J. C., Comellas, F., and Hsu, D. F. (1995). "distributed loop computer networks". *A survey Journal of Parallel and Distriputed Computing*, 24:2–10.

Bodlaender, H. L. (1989). "achromatic number is *NP*-complete for cographs and interval graphs". *Information Processing Letters*, 31:135–138.

Boesch, F. T. and Wang, J. (1985). "reliable circulant networks with minimum transmission delay". *IEEE Transactions on Circuit and systems*, 32:1286–1291.

Boutin, J. P., Duchene, E., Effantin, B., Kheddouci, H., and Seba, H. (2010). "new results about set colorings of graphs". *Electronic Journal of Combinatorics*, 17.

Cairnie, N. and Edwards, K. J. (1997). "some results on the achromatic number". *Journal of Graph Theory*, 26:129–136.

Cairnie, N. and Edwards, K. J. (1998). "achromatic number of bounded degree trees". *Discrete Mathematics*, 188:87–97.

Campbell, D. and Edwards, K. J. (2004). "a new lower bound for the harmonious chromatic number". *Australasian Journal of Combinatorics*, 29:99–102.

Casse, L. R., O'Keefe, C. M., and Wilson, B. J. (1994). "minimal harmoniously colorable designs". *Journal of Combinatorial Designs*, 2:61–69.

Chartrand, G. and L.Lesniak (2004). *"Graphs & Digraphs"*. A CRC Press Company.

Chartrand, G. and Zhang, P. (2009). *"Chromatic Graph Theory"*. A CRC Press Company.

Chaudhary, A. and Vishwanathan, S. (2001). "approximation algorithms for the achromatic number". *Journal of Algorithms*, 41:404–416.

Cichelli, R. J. (1980). "minimal perfect hash functions made simple". *Communications of the Association for Computing Machinery*, 23:17–19.

Edwards, K. (2009). *"A Bibliography of Harmonious Colourings and Achromatic Number"*. http://www.computing.dundee.ac.uk/staff/ kedwards/biblio.html.

Edwards, K. and McDiarmid, C. J. H. (1994). "new upper bounds on harmonious colorings". *Journal of Graph Theory*, 18:257–267.

Edwards, K. J. (1995). "the harmonious chromatic number of almost all trees". *Combinatorics, Probability and Computing*, 4:31–46.

Edwards, K. J. (1996). "the harmonious chromatic number of bounded degree trees". *Combinatorics, Probability and Computing*, 5:15–28.

Edwards, K. J. (1997). "the harmonious chromatic number of bounded degree graphs". *Journal of the London Mathematical Society(Series 2)*, 55:435–447.

Edwards, K. J. (1998). "a new upper bound for the harmonious chromatic number". *Journal of Graph Theory*, 29:257–261.

Elspas, B. and Turner, J. (1970). "graphs with circulant adjacency matrices". *Journal of Combinatorial Theory*, 9:297–307.

Euler, L. (1741). "*S*olutio problematis ad geometriam situs pertinentis". *Commentarii academiae scientiarum petropolitanae*, 8:128–140.

Farber, M., Hahn, G., Hell, P., and Miller, D. (1986). "concerning the achromatic number of graphs". *J. COMBINAT. THEORY SER. B.*, 40:21–39.

Frank, O., Harary, F., and Plantholt, M. J. (1982). "the line-distinguishing chromatic number of a graph". *Ars Combinatoria*, 14:241–252.

Franklin, J. N. (1974). "ambiguities in the x-ray analysis of crystal structures". *Acta. cryst.*, pages 698–702.

Gallian, J. A. (2011). "a dynamic survey of graph labeling". *Electronic Journal of Combinatorics*, 18:1–256.

Garrido, J. (1951). "le probleme de l' unicitedans la determination delstructures cristallines". *Bull Soc. Fr. Miner Crist*, 74:397–431.

Geller, D. P. and Kronk, H. V. (1974). "further results on the achromatic number". *Fundamenta Mathematicae*, 85:285–290.

Georges, J. P. (1995). "on the harmonious coloring of collections of graphs". *Journal of Graph Theory*, 20:241–254.

Halldorsson, M. M. (2004). *"An approximation algorithm for complete partition of regular graphs"*. http://www.hi.is/mmh/publications.html.

Harary, F. (1972). *"Graph Theory"*. Addison Wesley, Academic Press.

Harary, F. and Hedetniemi, S. T. (1970). "the achromatic number of a graph". *Journal of Combinatorial Theory*, 8:154–161.

Harary, F., Hedetniemi, S. T., and Prins, G. (1967). "an interpolation theorem for graphical homomorphisms". *Portugaliae Mathematica*, 26:453–462.

Hegde, S. M. (2009). "set colorings of graphs". *European Journal of Combinatorics*, 30:986–995.

Hell, P. and Miller, D. J. (1976). "graph with given achromatic number". *Discrete Mathematics*, 16:195–207.

Hell, P. and Miller, D. J. (1992). "achromatic numbers and graph operations". *Discrete Mathematics*, 108:297–305.

Hell, P. and Nesetril, J. (2004). *"Graphs and Homomorphisms"*. Oxford University Press.

Hopcroft, J. E. and Krishnamoorthy, M. S. (1983). "on the harmonious coloring of graphs". *SIAM Journal of Alg. Discrete Methods*, 4:306–311.

Kortsarz, G. and Krauthgamer, R. (2001). "on approximating the achromatic number". *SIAM Journal on Discrete Mathematics*, 14:408–422.

Kortsarz, G. and Shende, S. (2007). "an improved approximation of the achromatic number on bipartite graphs". *SIAM Journal on Discrete Mathematics*, 21:361–373.

Krasikov, I. and Roditty, Y. (1994). "bounds for the harmonious chromatic number of a graph". *Journal of Graph Theory*, 18:205–209.

Krysta, P. and Lorys, K. (2006). "efficient approximation algorithms for the achromatic number". *Theoretical Computer Science*, 361:150–171.

Kubale, M. (2004). *"Contemporary Mathematics, 352"*. American Mathematical Society,Providence,R.I.

Lee, S. M. and Mitchem, J. (1987). "an upper bound for the harmonious chromatic number of a graph". *Journal of Graph Theory*, 11:565–567.

Lewin, K. (1936). *"Principles of Topological Psychology"*. McGraw-Hill, New York.

Lu, Z. (1993). "the harmonious chromatic number of a complete binary and trinary tree". *Discrete Mathematics*, 118:165–172.

Lu, Z. (1995). "the harmonious chromatic number of a complete 4-ary tree". *Journal of Mathematical Research and Exposition*, 15:51–56.

Lu, Z. (1997). "the exact value of the harmonious chromatic number of a complete binary tree". *Discrete Mathematics*, 172:93–101.

Lu, Z. (1998). "exact value of the harmonious chromatic number of a complete trinary tree". *Systems Science and Mathematical Sciences*, 11:26–31.

MacGillivray, G. and Rodriguez, A. (2001). "the achromatic number of union of paths". *Discrete Mathematics*, 231:331–335.

Mammana, M. F. (2002). "on the harmonious chromatic number of $P(\alpha, K_n)$, $P(\alpha, K_{1,n})$ and $P(\alpha, K_{m,n})$". *Utilitas Mathematica*, 62:61–70.

Mate, A. (1981). "a lower estimate for the achromatic number of irreducible graphs". *Discrete Mathematics*, 33:171–183.

McDiarmid, C. J. H. and Xinhua, L. (1991). "upper bounds for harmonious colorings". *Journal of Graph Theory*, 15:629–636.

Miller, Z. and Pritikin, D. (1991). "the harmonious coloring number of a graph". *Discrete Mathematics*, 93:211–228.

Mitchem, J. (1989). "on the harmonious chromatic number of a graph". *Annals of Discrete Mathematics*, 74:151–157.

Patterson, A. L. (1944). "ambiguities in the x-ray analysis of crystal structures". *Phy. Rev.*, 64:195–201.

Roichman, Y. (2000). "on the achromatic number of hypercubes". *Journal of Combinatorial Theory, Series B*, 79:177–182.

Thilagavathi, K. and Roopesh, N. (2008). "achromatic colouring of central graphs and split graphs". *Far East Journal of Applied Mathematics*, 30:359–369.

Thilagavathi, K. and Roopesh, N. (2009). "generalisation of achromatic colouring of central graphs". *Electronic Notes in Discrete Mathematics*, 33:147–152.

Thilagavathi, K., Thilagavathi, K. P., and Roopesh, N. (2009). "the achromatic colouring of graphs". *Electronic Notes in Discrete Mathematics*, 33:153–156.

Thilagavathi, K. and Vivin, J. V. (2006). "harmonious colouring of graphs". *Far East J. Math. Sci. (FJMS)*, 20:189–197.

Thilagavathi, K., Vivin, J. V., and Anitha, B. (2007). "on harmonious coloring of line graph of central graphs of bipartite graphs". *Journal of Combinatorics, Information and System Sciences*, 32:232–240.

Vivin, J. V., Venkatachalam, M., and Ali, M. M. A. (2009). "a note on achromatic coloring of star graph families". *Filomat*, 23:251–255.

West, D. B. (2003). *"Introduction to Graph Theory"*. Prentice Hall of India, New Delhi.

Wilkov, R. S. (1972). "analysis and design of reliable computer networks". *IEEE Transactions on Communications*, 20:660–678.

Wong, G. K. and Coppersmith, D. A. (1974). "a combinatorial problem related to multimodule memory organization". *Journal of Association of Computing*, 23:392–401.

Xu, J. (2001). *"Topological Structure and Analysis of Interconnection Networks"*. Kluwer Academic Publishers.

Yannakakis, M. and Gavril, F. (1980). "edge dominating sets in graphs". *SIAM Journal on Applied Mathematics*, 38:364–372.

Publications

List of Publications /Communications Based on Thesis:

1. S. M. Hegde & Lolita Priya Castelino, *Further results on harmonious colorings of digraphs, AKCE International Journal of Graphs and Combinatorics*, 8, No.2 (2011), pp 151-159.

2. S. M. Hegde & Lolita Priya Castelino, *Harmonious colorings of digraphs,* accepted for publication in *ARS Combinatoria*, To appear.

3. S. M. Hegde & Lolita Priya Castelino, *Set Colorings of Digraphs,* accepted for publication in *Utilitas Mathematica*, To appear.

4. S. M. Hegde & Lolita Priya Castelino, *Achromatic Number of Some Classes of Digraphs,* revised version submitted to *Australasian Journal of Combinatorics.*

5. S. M. Hegde & Lolita Priya Castelino, *On Harmonious colorings of regular digraphs,* submitted to *AKCE International Journal of Graphs and Combinatorics.*

Bio-Data

Lolita Priya Castelino

Research Scholar,

Department of Mathematical and Computational Sciences,

National Institute of Technology Karnataka Surathkal,

P.O.Srinivasanagar,

Mangalore - 575025.

Phone: +919741968818.

Email: plolita@yahoo.com.

Permanent address

Mrs. Lolita Priya Castelino

W/0 Mr. Naval Castelino,

"Morning Star",

Subhasnagar Post,

Udupi Tq & Dist. -574137,

Karnataka.

Qualification

M.Sc. Mathematics, University of Madras, 2005.

Papers presented in the Conferences

1. S. M. Hegde and Lolita Priya Castelino, Harmonious Colorings of Digraphs, at *National Conference on Current Trends of Discrete Mathematics in Science and Social Systems* at Govinda Dasa College, Surathkal during $26^{th} - 28^{th}$ of February, 2009.

2. S. M. Hegde and Lolita Priya Castelino, Harmonious colorings of Digraphs, at *SIAM Conference on Discrete Mathematics* Austin, Texas during $14^{th} - 17^{th}$ of June, 2010.

3. S. M. Hegde and Lolita Priya Castelino, Harmonious Colorings of Oriented Paths and Oriented Cycles at *National Conference on Recent Trends in Computational Sciences and Engineering NCRTCSE-2011* at KVG College of Engineering, Sullia on 12th of February, 2011.

4. S. M. Hegde and Lolita Priya Castelino, Some results on the achromatic number of digraphs, at *Academy of Discrete Mathematics and Applications (ADMA)*, at NIT Calicut during $9^{th} - 11^{th}$ of June, 2011.