ON THE IMPLEMENTATION OF REGULARIZATION METHODS FOR NON-LINEAR ILL-POSED OPERATOR EQUATIONS

Thesis

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DOCTOR OF PHILOSOPHY

by

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MAY, 2014

DECLARATION

By the Ph.D. Research Scholar

I hereby *declare* that the Research Thesis entitled **ON THE IMPLEMENTATION OF REGULARIZATION METHODS FOR NONLINEAR ILL-POSED OPERATOR EQUATIONS** which is being submitted to the National Institute of Technology Karnataka, Surathkal in partial fulfillment of the requirements for the award of the Degree of **Doctor of Philosophy** in **Mathematical and Computational Sciences** is a *bonafide report of the research work carried out by me.* The material contained in this Research Thesis has not been submitted to any University or Institution for the award of any degree.

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CERTIFICATE

This is to *certify* that the Research Thesis entitled **ON THE IMPLEMENTATION OF REGULARIZATION METHODS FOR NONLINEAR ILL-POSED OPERATOR EQUATIONS** submitted by **Ms. SHUBHA V S**, (Register Number: 135007MA13F03) as the record of the research work carried out by her, is *accepted as the Research Thesis submission* in partial fulfillment of the requirements for the award of degree of **Doctor of Philosophy**.

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ABSTRACT OF THE THESIS

In this thesis we consider nonlinear ill-posed operator equations of the form $F(x) = y$, where $F : X \to Y$ is a nonlinear operator between Hilbert spaces X and Y. Many problems from computational sciences and other disciplines can be brought to the form $F(x) = y$. In practical applications, usually noisy data y^δ are available instead of y. The problem of recovery of the exact solution $\hat{x}$ from noisy equation $F(x) = y^\delta$ is ill posed, in the sense that a small perturbation in the data can cause large deviation in the solution and the solutions of these equations are usually unknown in the closed form. Thus the computation of a stable approximation for $\hat{x}$ from the solution of $F(x) = y^\delta$, becomes an important issue in the ill-posed problems, and most methods for solving these equations are iterative.

We consider iterative regularization methods and their finite dimensional realization, for obtaining an approximation for $\hat{x}$ in the Hilbert space. The choice of regularization parameter plays an important role in the convergence of regularization methods. We use the adaptive scheme of Pereverzev and Schock (2005), for choosing the regularization parameter. The error bounds obtained are of optimal order with respect to a general source condition.

Keywords: ***Ill-posed nonlinear equations, Regularization methods, Monotone operator, Lavrentive regularization, Tikhonov regularization, Projection methods, Adaptive method.***

Mathematics Subject Classification: 47J06, 47H30, 47H07, 49M15.

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Chapter 1

INTRODUCTION

1.1 GENERAL INTRODUCTION

Keller (1976) formulated the following general definition of inverse problems, which is frequently referred in the literature.

"We call two problems inverses of one another if the formulation of each involves all or part of the solution of the other. Direct problems has been studied widely for some time, while the other is newer and not so well understood and it is called the inverse problem."

Inverse problems are wide in range, and are important in applied mathematics & other sciences which have witnessed a rapid growth over past few decades. Inverse problems have wide variety of applications in sciences and engineering. A well known and prominently known real world medical application includes tomography, cell detection in various cancer diseases, which helps to calculate the defective cell densities in human body.

In many practical applications it is observed that, the inverse problems are not well-posed, in the sense that a unique solution that continuously depends on data is not guaranteed. A problem which is not well-posed is called ill-posed(see Definition 1.2.1.)

If the range space is defined as set of solutions to the direct or forward problem, existence of a solutions to the inverse problem is clear. However, a solution may fail to exist if the elements of range space are perturbed by noise. Uniqueness of solution to an inverse problem is often not easy to show, but it is an important issue. If the uniqueness is not guaranteed by the given data, then either additional data have to be observed or the set of favorable solutions has to be restricted using a-priori information on the solution.

Until the beginning of the last century it was generally believed that for natural problems the solution will depend continuously on the data. If this was not the case, the mathematical model of the problem was believed to be inadequate. Therefore, these problems were called ill or badly posed. Only in the second half of the last century it was realized that a large number of problems arising in science and technology are ill-posed in any reasonable mathematical setting. This initiated a large amount of research in stable and accurate methods for numerical solution of ill-posed problems. Today inverse and ill-posed problems are still an active area of research. This is reflected in a large number of article in the journals("Inverse Problems", "Inverse Problems and Imaging", "Inverse and Ill-Posed Problems", "Inverse Problems in Engineering", "Mathematical Inverse Problems" etc) and monographs(see for example Engl et al. (1996), Groetsch (1993), Keller (1976), Ramm (2005)).

1.2 DEFINITIONS AND BASIC RESULTS

Let X be a real Hilbert space with inner product $\langle ., . \rangle$ and norm $\|.\|$. Let $B(x, r)$ and $\overline{B(x, r)}$, stand respectively, for the open and closed balls in X with center x and radius $r > 0$.

DEFINITION 1.2.1. *Let X and Y are Hilbert spaces and $F : X \to Y$ be an operator(linear or non-linear). Then the equation*

$$F(x) = y \tag{1.2.1}$$

is said to be well-posed if the following three conditions hold.

a) for each $y \in Y$, there is a solution $x \in X$ of (1.2.1),

b) the solution x is unique in X and

c) the dependence of x upon y is continuous.

If condition (a) and (b) holds then the map is invertible and condition (c) indicates that the inverse mapping is continuous. An equation of the type (1.2.1) which is not well-posed is called ill-posed.

It is always assumed that (1.2.1) has a solution $\hat{x}$, for exact data, that is, $F(\hat{x}) = y$, but due to non-linearity of F(or if F is not injective), this solution need not be unique. Therefore we consider a x_0-minimal norm solution of (1.2.1). Recall that (Engl et al. (1996), Tautenhahn and Jin (2003), Qi-nian and Zong-yi (1999)) a solution $\hat{x}$ of (1.2.1) is said to be an x_0-minimal norm solution (x_0-MNS) of (1.2.1) if

$$F(\hat{x}) = y, \tag{1.2.2}$$

$$\|x_0 - \hat{x}\| = \min_{x \in D(F)} \{\|x - x_0\| : F(x) = y\}. \tag{1.2.3}$$

Here and below $D(F)$ denote the domain of F. The elements $x_0 \in X$ in (1.2.3) plays the role of a selection criteria (Engl et al. (1989)) and it is assumed to be known.

We need the following definitions in the sequel.

DEFINITION 1.2.2. *Let F be an operator mapping a Hilbert space X into a Hilbert space Y. If there exists a bounded linear operator $L : X \to Y$ such that for $x_0 \in X$*

$$\lim_{h \to 0} \frac{\|F(x_0 + h) - F(x_0) - L(h)\|}{\|h\|} = 0,$$

then F is said to be a Fréchet-differentiable at x_0 and the bounded linear operator $F'(x_0) := L$ is called the first Fréchet-derivative of F at x_0.

DEFINITION 1.2.3. *$F : X \to X$ is a monotone operator if it satisfies the relation*

$$\langle F(v) - F(w), v - w\rangle \geq 0, \quad \forall v, w \in D(F).$$

REMARK 1.2.4. *(Ramm, 2005). If F is a linear operator, then problem (1.2.1) is ill-posed if either $N(F) \neq 0$ or $y \notin R(F)$ or $R(F)$ is not closed ($N(F)$ is the null space of F and $R(F)$ is the range of F). i.e., F^{-1} is unbounded. For a nonlinear and Fréchet differentiable operator F there* *are several possibilities of ill-posedness. If $F'(x)$ is continuously invertible at some point x, then $F(x)$ is a local homeomorphism. But F may be a homeomorphism, despite of the fact that $F'(x)$ is not continuously invertible.*

In nonlinear case, ill-posedness always means that the solutions do not depend continuously on the data i.e., $F'(x)$ is not continuously invertible.

The theory of linear ill-posed problems is very well-developed (Engl et al., 1996) and can be considered as almost complete. Hence, we deal with non-linear case, where the theory is not so well developed as in the case of linear one. Next we give few motivational examples of ill-posed problems.

EXAMPLE 1.2.5. Geological Prospecting:*(Vasin and George (2014) and Vasin et al. (1996)). Let the half-space is modeled by two-layers of constant different densities σ_1, σ_2 separated by a surface S to be determined. In the Cartesian coordinate system, whose plane xOy coincides with the ground surface and the axis z is directed downward, the inverse gravimetry problem has the form (see, Vasin et al. (1996) and references in it):*

$$\Gamma\Delta\sigma \int\int_D \frac{1}{[(x-x')^2+(y-y')^2+H^2]^{1/2}}dx'dy'$$
$$-\int\int_D \frac{1}{[(x-x')^2+(y-y')^2+u^2(x',y')]^{1/2}}dx'dy' = \Delta g(x,y); \tag{1.2.4}$$

here Γ is gravity constant, $\Delta\sigma = \sigma_1 - \sigma_2$ is the density jump at the interface S, described by the function $u(x,y)$ to be evaluated. $\Delta g(x,y)$ is the anomalous gravitational field caused by deviation of the interface S from horizontal asymptotic plane $z = H$, i.e., for the sought for solution $\hat{u}(x,y)$ the following relation holds

$$\lim_{|x|,|y|\to\infty} |\hat{u}(x,y) - H| = 0,$$

$g(x,y)$ is given on the domain D.

Since in (1.2.4) the first term does not depend on $u(x,y)$ equation can be written as

$$F(u) \equiv -\int\int_D \frac{1}{[(x-x')^2+(y-y')^2+u^2(x',y')]^{1/2}}dx'dy' = f(x,y),$$

where $f(x,y) = \Delta g(x,y) + F(H)$.

EXAMPLE 1.2.6. *(Nguyen, 1998). Non-linear singular equation in the form*

$$\int_0^t (t-s)^{-\lambda} x(s)ds + F(x(t)) = f_0(t), 0 < \lambda < 1 \tag{1.2.5}$$

where $f_0 \in L^2[0,1]$ *and non-linear function* $F(t)$ *satisfies the following conditions:*

1) $\|F(t)\| \leq a_1 + a_2|t|, a_1, a_2 > 0$

2) $F(t_1) \leq F(t_2)$ *iff* $t_1 \leq t_2$

3) F *is differentiable.*

Thus, F *is a monotone operator from* $X = L^2[0,1]$ *into* $Y = L^2[0,1]$*. In addition, assume that* F *is compact operator. Then the equation (1.2.5) is an ill-posed problem because the operator* K *defined by*

$$Kx(t) = \int_0^t (t-s)^{-\lambda} x(s)ds,$$

is also compact.

EXAMPLE 1.2.7. *(Hoang and Ramm, 2010). Consider a nonlinear operator equation* $F : L^2[0,1] \to L^2[0,1]$ *defined by*

$$F(u) := B(u) + (\arctan(u))^3 := \int_0^1 e^{-|x-y|} u(y)dy + (\arctan(u))^3. \tag{1.2.6}$$

Since the function $u \to \arctan^3 u$ *is an increasing function on* $\mathbb{R}$*, one has*

$$\langle (\arctan(u))^3 - (\arctan(v))^3, u - v \rangle \geq 0, \forall u, v \in L^2[0,1].$$

Moreover,

$$e^{-|x|} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{1+\lambda^2} d\lambda$$

Therefore, $\langle B(u-v), u-v\rangle \geq 0$, *so*

$$\langle F(u-v), u-v\rangle \geq 0, \forall u, v \in L^2[0,1].$$

Thus, F is a monotone operator. Note that

$$\langle (\arctan(u))^3 - (\arctan(v))^3, u-v\rangle = 0 \quad \text{iff} \quad u = v \quad \text{a.e.}$$

Therefore, the operator F, defined in (1.2.6), is injective and F has at most one solution. The Fréchet derivative of F is

$$F'(u)w = \frac{3(arctan(u))^2}{1+u^2}w + \int_0^1 e^{-|x-y|}w(y)dy.$$

If $u(x)$ vanishes on a set of positive Lebesgue measure, then $F'(u)$ is not boundedly invertible. If $u \in C[0,1]$ vanishes even at one point x_0, then $F'(u)$ is not boundedly invertible in $L^2[0,1]$.

1.3 REGULARIZATION METHODS

Since (1.2.1) is ill-posed, regularization techniques are required to obtain an approximation for $\hat{x}$. By regularization method of ill-posed equation, (1.2.1) with y^δ in place of y, where $\|y - y^\delta\| \leq \delta$, we mean, a family $\{R_\alpha : \alpha > 0\}$ of bounded linear operator from Y to X such that, $R_\alpha y^\delta \to \hat{x}$ as $\alpha \to 0$ and $\delta \to 0$.

1.3.1 Tikhonov regularization

Tikhonov Regularization has been investigated by many authors (see e.g Engl et al. (1996), Engl et al. (1989) and Neubauer (1989)) to solve non-linear ill-posed problems in a stable manner. In Tikhonov Regularization, a solution of the problem (1.2.1) is approximated by a solution of the minimization problem

$$J_\alpha(x) = \min_{x \in D(F)} \|F(x) - y^\delta\|^2 + \alpha\|x - x_0\|^2 \qquad (1.3.1)$$

where $\alpha > 0$ is a small regularization parameter and $y^\delta \in Y$ is the available noisy data, for which we have the additional information that

$$\|y - y^\delta\| \leq \delta.$$

It is known that (Engl et al., 1989) the minimizer x_α^δ of the functional $J_\alpha(x)$ satisfies the Euler equation

$$F'(x)^*(F(x) - y^\delta) + \alpha(x - x_0) = 0.$$

Here $F'(.)^* : Y \to X$ denotes the adjoint of the Fréchet derivative $F'(.) : X \to Y$. It is also known that (Engl et al., 1996) for properly chosen regularization parameter α, the minimizer x_α^δ of the functional $J_\alpha(x)$ is a good approximation to the solution $\hat{x}$ with minimal distance from x_0. Thus the main focus is to find a minimizing element x_α^δ of the Tikhonov functional (1.3.1). But the Tikhonov functional with non-linear operator F might have several minima, so to ensure the convergence of any optimization algorithm to a global minimizer x_α^δ of the Tikhonov functional (1.3.1), one has to employ stronger restrictions on the operator (Ramlau, 2003).

1.3.2 Lavrentiev regularization

In this section we assume that $X = Y$ is a real Hilbert space. When F is a nonlinear monotone operator, instead of Tikhonov Regularization method, one may consider Lavrentiev regularization method. In this method the regularized approximation x_α^δ is obtained by solving the operator equation

$$F(x) + \alpha(x - x_0) = y^\delta \tag{1.3.2}$$

where x_0 is an initial guess of $\hat{x}$. Using Minty-Browder Theorem (see section 2.1 Alber and Ryazantseva (2006)) one can prove that (1.3.2) has a unique solution for all $\alpha > 0$.

1.3.3 Choice of regularization parameter

In general a regularized solution x_α^δ can be written as $x_\alpha^\delta = R_\alpha y^\delta$, where R_α is a regularization function. A regularization method consists not only of a choice of regularization function R_α but also of a choice of the regularization parameter α. A choice $\alpha = \alpha_\delta$ of the regularization parameter may be made in either an a priori or a posteriori way (Groetsch (1993)). Suppose there exists a function $\varphi : (0, a] \to (0, \infty)$ with $a \geq \|F'(.)\|$ and $v \in X$ such that

$$x_0 - \hat{x} = \varphi(F'(.))v, \tag{1.3.3}$$

where x_0 is an initial guess and $F'(\hat{x})$ is the Fréchet derivative of F at some point, and

$$\|\hat{x} - R_\alpha y\| \leq \varphi(\alpha),$$

then φ is called a source function and the condition (1.3.3) is called source condition.

Note that (Groetsch (1993)) the choice of the parameter α_δ depends on the unknown source conditions. In applications, it is desirable that α is chosen independent of the source function φ, but may depend on the data (δ, y^δ), and consequently on the regularized solutions. For linear ill-posed problems there exist many such a posteriori parameter choice strategies (George and Nair (1993), Groetsch and Guacaneme (1987), Guacaneme (1990) and Tautenhahn (2002)).

Pereverzev and Schock (2005), considered an adaptive selection of the parameter which does not involve even the regularization method in an explicit manner. Let us briefly discuss this adaptive method in a general context of approximating an element $\hat{x} \in X$ by elements from the set $\{x_\alpha^\delta : \alpha > 0, \delta > 0\}$.

Assume that there exist increasing functions $\varphi(t)$ and $\psi(t)$ for $t > 0$ such that

$$\lim_{t\to 0} \varphi(t) = 0 = \lim_{t \to 0} \psi(t)$$

and

$$\|\hat{x} - x_\alpha^\delta\| \leq \varphi(\alpha) + \frac{\delta}{\psi(\alpha)},$$

for all $\alpha > 0, \delta > 0$. Here, the function φ may be associated with the unknown element $\hat{x}$, whereas the function ψ may be related to the method involved in obtaining x_α^δ. Note that the quantity $\varphi(\alpha) + \frac{\delta}{\psi(\alpha)}$ attains its minimum for the choice $\alpha := \alpha_\delta$ such that $\varphi(\alpha_\delta) = \frac{\delta}{\psi(\alpha_\delta)}$, that is for

$$\alpha_\delta = (\varphi\psi)^{-1}(\delta)$$

and in that case

$$\|\hat{x} - x_\alpha^\delta\| \leq 2\varphi(\alpha_\delta).$$

The above choice of the parameter is a priori in the sense that it depends on unknown functions φ and ψ.

In an a posteriori choice, one finds a parameter α_δ without making use of the unknown source function φ such that one obtains an error estimate of the form

$$\|\hat{x} - x_{\alpha_\delta}\| \leq c\varphi(\alpha_\delta),$$

for some $c > 0$ with $\alpha_\delta = (\varphi\psi)^{-1}(\delta)$. The procedure in Pereverzev and Schock (2005) starts with a finite number of positive real numbers, $\alpha_0, \alpha_1, \alpha_2, ..., \alpha_N$, such that $\alpha_0 < \alpha_1 < \alpha_2 < ... < \alpha_N$. The following theorem is essentially a reformulation of a theorem proved in Pereverzev and Schock (2005).

THEOREM 1.3.1. *(see, George and Nair (2008)) Assume that there exists $i \in \{0, 1, 2, ..., N\}$ such that $\varphi(\alpha_i) \leq \frac{\delta}{\psi(\alpha_i)}$ and for some $\mu > 1$,*

$$\psi(\alpha_i) \leq \mu\psi(\alpha_{i-1}) \quad \forall i \in \{0, 1, 2, ...N\}.$$

Let

$$l := \max\{i : \varphi(\alpha_i) \leq \frac{\delta}{\psi(\alpha_i)}\} < N$$

$$k := \max\{i : \|x_{\alpha_i}^\delta - x_{\alpha_j}^\delta\| \leq 4\frac{\delta}{\psi(\alpha_j)}, \forall j = 0, 1, ..., i-1\}.$$

Then $l \leq k$ and

$$\|\hat{x} - x^{\delta}_{\alpha_k}\| \leq 6\mu\varphi(\alpha_\delta), \alpha_\delta := (\varphi\psi)^{-1}(\delta).$$

1.3.4 Iterative methods

In the last few years many authors considered iterative methods, for example, Landweber method (Hanke et al. (1995), Ramlau (1999)), Levenberg-Marquardth method (Hanke (1997a)), Gauss-Newton (Bakushinskii (1992), Blaschke et al. (1997)), Conjugate Gradient (Hanke (1997b)) and Newton like methods (Kaltenbacher (1997), Nair and Ravishankar (2008)). Iterative methods for finding the solution of $F(x) = 0$ have the following form:

1) Beginning with a starting value x_0,

2) Successive approximates $x_i, i = 1, 2, ...$ to x^* are computed with the aid of an iteration function $G : X \to X$, defined as $G(x_i) = x_{i+1}, i = 1, 2, ...,$

3) If x^* is a fixed point of G i.e., $G(x^*) = x^*$, all fixed points of G are also zeros of F, and if G is continuous in a neighbourhood of each of its fixed points, then each limit point sequence $x_i, i = 1, 2...$, is a fixed point of G, and hence a solution of the equation $F(x) = 0$.

Most of the above mentioned iterative methods involves inverse and Fréchet derivative of the operator F. Ramlau (2003), considered a method called TIGRA (Tikhonov gradient method) for the implementation of Tikhonov Regularization defined iteratively by

$$x^{\delta}_{k+1} = x^{\delta}_k + \beta_k[F'(x^{\delta}_k)^*(y^{\delta} - F(x^{\delta}_k)) + \alpha_k(x^{\delta}_k - x_0)]. \qquad (1.3.4)$$

Note that the above method is free from the inverse of the Fréchet derivative of the operator involved.

In this thesis we considered iterative regularization method involving inverse of the Fréchet derivative of the operator under consideration and also we considered an iterative method free from the inverse of the Fréchet derivative of the operator.

1.4 OUTLINE OF THE THESIS

The thesis is organized in six chapters. In Chapter 1 we present general introduction to the problem, some examples for nonlinear ill-posed equations, introduction to some regularization methods and some results related to the thesis.

In Chapter 2, for solving $F(x) = y$, we consider an iterative method defined for $n = 1, 2, 3, \cdots$ by

$$x^{\delta}_{n+1,\alpha} = x^{\delta}_{n,\alpha} - (F'(x^{\delta}_{n,\alpha}) + \alpha I)^{-1}[F(x^{\delta}_{n,\alpha}) - y^{\delta} + \alpha(x^{\delta}_{n,\alpha} - x_0)] \qquad (1.4.5)$$

where $x_0 := x^{\delta}_{0,\alpha}$ is the starting point of the iteration. We make use of the adaptive scheme suggested by Pereverzev and Schock (2005) for choosing the regularization parameter α, depending on the noisy data y^{δ} and the error δ. Under general source condition on $x_0 - \hat{x}$, the error $\|x^{\delta}_{n,\alpha} - \hat{x}\|$ between the regularized approximation $x^{\delta}_{n,\alpha}$ and the exact solution $\hat{x}$ is of optimal order.

Chapter 3 deals with finite dimensional realization of the method considered in Chapter 2. The algorithm for the proposed method is given followed by a numerical example which confirms the efficiency of our approach.

Chapter 4 is the modified Tikhonov Gradient-Type method(TGTM) is defined for $n = 1, 2, 3, ...$ by

$$u^{\delta}_{n+1,\alpha} = u^{\delta}_{n,\alpha} - \beta(F'(u^{\delta}_{n,\alpha})^{*}(F(u^{\delta}_{n,\alpha}) - y^{\delta}) + \alpha(u^{\delta}_{n,\alpha} - u_0)) \qquad (1.4.6)$$

where $u_0 = u^{\delta}_{0,\alpha}$ is the initial approximation. Note that inversion of the Fréchet derivative of the operator is not involved in the above method. We select the regularization parameter α using adaptive method and obtained an optimal order estimate.

In Chapter 5, we consider finite dimensional realization of the method considered in Chapter 4. Numerical examples and corresponding computational results are presented.

In Chapter 6, we conclude the thesis by giving scope for future work.

Chapter 2

A QUADRATIC CONVERGENCE YIELDING ITERATIVE METHOD FOR THE IMPLEMENTATION OF LAVRENTIEV REGULARIZATION METHOD FOR ILL-POSED EQUATIONS

George and Elmahdy (2012), considered an iterative method which converges quadratically to the unique solution x_α^δ of the method of Lavrentiev regularization, i.e., $F(x) + \alpha(x - x_0) = y^\delta$, approximating the solution $\hat{x}$ of the ill-posed problem $F(x) = y$ where $F : D(F) \subseteq X \longrightarrow X$ is a nonlinear monotone operator defined on a real Hilbert space X. The convergence analysis of the method was based on a majorizing sequence. In this chapter, we expand the applicability of the method considered by George and Elmahdy (2012) by weakening the restrictive conditions imposed on the radius of the convergence ball and also by weakening the popular Lipschitz-type hypotheses considered in earlier studies such as George and Elmahdy (2012), Mahale and Nair (2009), Nair and Ravishankar (2008), Semenova (2010) and Tautenhahn (2002). We show that the adaptive scheme considered by Pereverzev and Schock (2005) for choosing the regularization parameter can be effectively used here for obtaining optimal order error estimate.

2.1 INTRODUCTION

For monotone operator F, one usually uses the Lavrentiev regularization method (Tautenhahn (2002)). In this method the regularized approximation x_α^δ is obtained by solving the operator equation

$$F(x) + \alpha(x - x_0) = y^\delta. \tag{2.1.1}$$

It is known (cf. Tautenhahn (2002),Theorem 1.1) that (2.1.1) has unique solution x_α^δ for $\alpha \geq 0$, provided F Fréchet differentiable and monotone in the ball $B(\hat{x}, r) \subset D(F)$ with radius $r = \| \hat{x} - x_0 \| + \frac{\delta}{\alpha}$. (In section 2.2 we prove that (2.1.1) has a unique solution for all $x \in B(x_0, r)$ under some assumption on the Fréchet derivative of F). However the regularized equation (2.1.1) remains nonlinear and one may have difficulties in solving it numerically. So one has to use iterative regularization methods (Bakushinsky and Smirnova (2005), Blaschke et al. (1997), Deuflhard et al. (1998), George (2006), George (2010), George and Nair (2008), Hoang and Ramm (2010), Mahale and Nair (2009), Nair and Ravishankar (2008), Ortega and Rheinboldt (1970), Semenova (2010), Tautenhahn (2002)).

George and Elmahdy (2012), considered the method, defined iteratively for $n = 1, 2, 3, \cdots$ by

$$x_{n+1,\alpha}^\delta = x_{n,\alpha}^\delta - (F'(x_{n,\alpha}^\delta) + \alpha I)^{-1}[F(x_{n,\alpha}^\delta) - y^\delta + \alpha(x_{n,\alpha}^\delta - x_0)], \tag{2.1.2}$$

where $x_0 := x_{0,\alpha}^\delta$ is the starting point of the iteration for solving (2.1.1). They proved that $x_{n,\alpha}^\delta$ converges quadratically to x_α^δ. The convergence analysis in George and Elmahdy (2012), was based on a majorizing sequence and the conditions (see(2.10) and (2.11) in George and Elmahdy (2012)) required for the convergence of the method are not easy to verify. The convergence analysis in George and Elmahdy (2012) was carried out using the following assumptions:

ASSUMPTION 2.1.1. *There exists $r > 0$ such that $B(x_0, r) \cup B(\hat{x}, r) \subset D(F)$ and F is Fréchet differentiable at all $x \in B(x_0, r) \cup B(\hat{x}, r)$.*

ASSUMPTION 2.1.2. *There exists a constant* $K_0 > 0$ *such that for every* $u, v \in B(x_0, r) \cup B(\hat{x}, r)$ *and* $w \in X$*, there exists an element* $\phi(u, v, w) \in X$ *satisfying* $[F'(u) - F'(v)]w = F'(v)\phi(u, v, w)$*,* $\|\phi(u, v, w)\| \leq K_0\|w\|\|u - v\|$*.*

ASSUMPTION 2.1.3. *There exists a continuous and strictly increasing function* $\varphi : (0, a] \to (0, \infty)$ *with* $a \geq \|F'(\hat{x})\|$ *satisfying;*

(i) $lim_{\lambda \to 0}\varphi(\lambda) = 0$,

(ii) $sup_{\lambda \geq 0}\frac{\alpha\varphi(\lambda)}{\lambda+\alpha} \leq c_\varphi\varphi(\alpha) \qquad \forall\alpha \in (0, a]$ *and*

(iii) there exists $v \in X$ *with* $\|v\| \leq 1$ *such that*

$$x_0 - \hat{x} = \varphi(F'(\hat{x}))v.$$

But in our result, we replace Assumption 2.1.3 by the following.

ASSUMPTION 2.1.4. *There exists a continuous and strictly increasing function* $\varphi : (0, a] \to (0, \infty)$ *with* $a \geq \|F'(x_0)\|$ *satisfying;*

(i) $lim_{\lambda \to 0}\varphi(\lambda) = 0$,

(ii) $sup_{\lambda \geq 0}\frac{\alpha\varphi(\lambda)}{\lambda+\alpha} \leq \varphi(\alpha) \qquad \forall\alpha \in (0, a]$ *and*

(iii) there exists $v \in X$ *with* $\|v\| \leq 1$ *such that*

$$x_0 - \hat{x} = \varphi(F'(x_0))v. \tag{2.1.3}$$

We replace Assumption 2.1.2 by the following.

ASSUMPTION 2.1.5. *Suppose there exists a constant* $K_0 > 0$ *such that for all* $w \in X$ *and* $u, v \in B(x_0, r) \subseteq D(F)$*, there exists element* $\phi(u, v, w) \in X$ *such that* $[F'(u) - F'(v)]w = F'(v)\phi(u, v, w)$*,* $\| \phi(u, v, w)\| \leq K_0\|w\|\|u - v\|$*.*

Using the above assumptions, we prove that the method (2.1.2) converges quadratically to the solution x_α^δ of (2.1.1).

Note that, a sequence (x_n) is said to be converging quadratically to x^*, if there exists a positive number M_q, not necessarily less than 1, such that

$$\|x_{n+1} - x^*\| \leq M_q \|x_n - x^*\|^2,$$

for all n sufficiently large. And the convergence of (x_n) to x^*, is said to be linear if there exists a positive number $M_0 \in (0, 1)$, such that

$$\|x_{n+1} - x^*\| \leq M_0 \|x_n - x^*\|.$$

Note that regardless of the value of M_q quadratic convergent sequence will always eventually converge faster than a linear convergent sequence. For an extensive discussion of convergence rate see Ortega and Rheinboldt (1970). We provide an optimal order error estimate under a general source condition on $x_0 - \hat{x}$. We choose the regularization parameter α in $(x^\delta_{n,\alpha})$ from finite set

$$D_N := \{\alpha_i : 0 < \alpha_0 < \alpha_1 < \cdots < \alpha_N\},$$

using the adaptive parameter selection procedure suggested by Pereverzev and Schock (2005).

Further we replace Assumption 2.1.5 by the weaker assumption:

ASSUMPTION 2.1.6. *There exists a constant $k_0 > 0$ such that for all $w \in X$ and $u \in B(x_0, r)$, there exists an element, say $\phi(u, x_0, w) \in X$ satisfying $[F'(u) - F'(x_0)]w = F'(x_0)\phi(u, x_0, w)$ and $\|\phi(u, x_0, w)\| \leq k_0 \|w\| \|u - x_0\|$.*

REMARK 2.1.7. *There are classes of operators for which Assumption 2.1.5 is not satisfied but the method (2.1.2) converges.*

Here we extend the applicability of (2.1.2) with the following advantages:

(1) The sufficient convergence criteria are weaker.

(2) The computational cost of the constant k_0 is smaller than that of the constant K_0, even when $K_0 = k_0$.

(3) The convergence domain of (2.1.2) with Assumption 2.1.6 can be large, since $\frac{K_0}{k_0}$ can be arbitrarily small(see Example 4.4 in (Argyros and George, 2013)).

(4) Note that Assumption 2.1.3 involves the Fréchet derivative at the exact solution $\hat{x}$ which is unknown in practice, while Assumption 2.1.4 depends on the Fréchet derivative F at x_0.

(5) Assumption 2.1.6 is weaker than Assumption 2.1.5(see Example 4.3 in Argyros and George (2013)).

2.2 PREPARATORY RESULTS

First we prove that (2.1.1) has a unique solution x_α^δ in $B(x_0, r)$.

THEOREM 2.2.1. *Let $\hat{x} \in B(x_0, r) \subset D(F)$ be a solution of (1.2.1). Assumption 2.1.5(or Assumptions 2.1.6) is satisfied, and let $F := D(F) \subseteq X \longrightarrow X$ be Fréchet differentiable in $B(x_0, r)$ with $K_0 r < 1$ (or $k_0 r < 1$). Then the regularized problem (2.1.1) possesses a unique solution x_α^δ in $B(x_0, r)$.*

Proof. For $x \in B(x_0, r)$, let $M_x = \int_0^1 (F'(\hat{x} + t(x - \hat{x})))dt$. Suppose $M_x + \alpha I$ is invertible. Then

$$(M_x + \alpha I)(x - \hat{x}) = \alpha(x_0 - \hat{x}) + y^\delta - y \tag{2.2.1}$$

has a unique solution x_α^δ in $B(x_0, r)$. Thus $F(x) - y^\delta + \alpha(x - x_0) = (M_x + \alpha I)(x - \hat{x}) - \alpha(x_0 - \hat{x}) - (y^\delta - y) = 0$ has a unique solution x_α^δ in $B(x_0, r)$. So it remains to show that for $K_0 r < 1$(or $k_0 r < 1$), $M_x + \alpha I$ is invertible.

Let $A_0 := F'(x_0)$. Note that by Assumption 2.1.5(or 2.1.6), we have

$$\begin{aligned}
\|(A_0+\alpha I)^{-1}(M_x - A_0)\| &= \sup_{\|v\|\le 1} \|(A_0+\alpha I)^{-1}(M_x - A_0)v\| \\
&= \sup_{\|v\|\le 1} \|(A_0+\alpha I)^{-1}\int_0^1 [F'(\hat{x}+t(x-\hat{x})) \\
&\quad -F'(x_0)]vdt\| \\
&= \sup_{\|v\|\le 1} \|(A_0+\alpha I)^{-1}A_0 \int_0^1 \phi(\hat{x}+t(x-\hat{x}), x_0, v)dt\| \\
&\le \int_0^1 \|\phi(\hat{x}+t(x-\hat{x}), x_0, v)dt\| \le K_0 r(\text{or } k_0 r) < 1.
\end{aligned}$$

So $I+(A_0+\alpha I)^{-1}(M_x - A_0)$ is invertible for all $x \in B(x_0, r)$ with $K_0 r < 1$(or $k_0 r < 1$). Now from the relation $M_x + \alpha I = (A_0+\alpha I)[I+(A_0+\alpha I)^{-1}(M_x - A_0)]$, it follows that for all $x \in B(x_0, r)$ with $K_0 r < 1$, (2.1.1) has a solution in $B(x_0, r)$. □

THEOREM 2.2.2. *Let* $0 < K_0 r(\text{or } k_0 r) < 1, x_\alpha^\delta$ *be the solution of (2.1.1),* $\hat{x} \in B(x_0, r)$*, Assumption 2.1.4 and Assumption 2.1.5 (or 2.1.6) be satisfied. Then*

$$\|x_\alpha^\delta - \hat{x}\| \le \frac{1}{1-K_0 r}\left(\frac{\delta}{\alpha} + \varphi(\alpha)\right). \tag{2.2.2}$$

Proof. Let $M := \int_0^1 F'(\hat{x}+t(x_\alpha^\delta - \hat{x}))dt$. Then by fundamental theorem of integration

$$F(x_\alpha^\delta) - F(\hat{x}) = M(x_\alpha^\delta - \hat{x}) \tag{2.2.3}$$

and hence by (2.1.1), we have

$$(M+\alpha I)(x_\alpha^\delta - \hat{x}) = y^\delta - y + \alpha(x_0 - \hat{x}).$$

Thus

$$\begin{aligned}
x_\alpha^\delta - \hat{x} &= (A_0+\alpha I)^{-1}[y^\delta - y + \alpha(x_0-\hat{x}) + (A_0 - M)(x_\alpha^\delta - \hat{x})] \\
&= s_1 + s_2 + s_3
\end{aligned}$$

where $s_1 := (A_0+\alpha I)^{-1}(y^\delta - y), s_2 := (A_0+\alpha I)^{-1}\alpha(x_0 - \hat{x})$ and $s_3 :=$

$(A_0 + \alpha I)^{-1}(A_0 - M)(x_\alpha^\delta - \hat{x})$. Note that

$$\|s_1\| \leq \frac{\delta}{\alpha} \tag{2.2.4}$$

by Assumption 2.1.4

$$\|s_2\| \leq \varphi(\alpha) \tag{2.2.5}$$

and by Assumption 2.1.5(or 2.1.6), we have

$$\begin{aligned} \|s_3\| &= \|(A_0 + \alpha I)^{-1}(A_0 - M)(x_\alpha^\delta - \hat{x})\| \\ &= \|(A_0 + \alpha I)^{-1} \int_0^1 [F'(x_0) - F'(\hat{x} + t(x_\alpha^\delta - \hat{x}))](x_\alpha^\delta - \hat{x})\| \\ &= \|(A_0 + \alpha I)^{-1} A_0 \int_0^1 \phi(\hat{x} + t(x_\alpha^\delta - \hat{x}), x_0, x_\alpha^\delta - \hat{x}) dt\| \\ &\leq K_0 r \|x_\alpha^\delta - \hat{x}\|. \end{aligned} \tag{2.2.6}$$

The result now follows from (2.2.4)-(2.2.6). □

2.3 CONVERGENCE ANALYSIS OF (2.1.2) USING ASSUMPTION 2.1.5

In this section we prove that $x_{n,\alpha}^\delta$ converges to x_α^δ quadratically using Assumption 2.1.5.

THEOREM 2.3.1. *Suppose Assumption 2.1.5 holds and* $r < \min\{\frac{1}{K_0}, 1\}$. *Let* $x_{n+1,\alpha}^\delta$ *be as in (2.1.2). Then* $x_{n,\alpha}^\delta$ *converges quadratically to* x_α^δ *and*

$$\|x_{n+1,\alpha}^\delta - x_\alpha^\delta\| \leq (\frac{K_0}{2})^{2^{n+1}-1} e^{-\gamma_r 2^{n+1}} \tag{2.3.1}$$

where $\gamma_r = -\ln r > 0$.

Proof. Since $F(x_\alpha^\delta) + \alpha(x_\alpha^\delta - x_0) = y^\delta$, we have

$$x_{n+1,\alpha}^\delta - x_\alpha^\delta \;=\; x_{n,\alpha}^\delta - x_\alpha^\delta - (F'(x_{n,\alpha}^\delta) + \alpha I)^{-1}[F(x_{n,\alpha}^\delta) - F(x_\alpha^\delta) + \alpha(x_{n,\alpha}^\delta - x_\alpha^\delta)]$$

$$
\begin{aligned}
&= (F'(x^\delta_{n,\alpha}) + \alpha I)^{-1}[F'(x^\delta_{n,\alpha})(x^\delta_{n,\alpha} - x^\delta_\alpha) \\
&\quad -(F(x^\delta_{n,\alpha}) - F(x^\delta_\alpha)] \\
&= (F'(x^\delta_{n,\alpha}) + \alpha I)^{-1}[F'(x^\delta_{n,\alpha}) - \int_0^1 F'(x^\delta_\alpha + t(x^\delta_{n,\alpha} - x^\delta_\alpha))dt] \\
&\quad (x^\delta_{n,\alpha} - x^\delta_\alpha) \\
&= (F'(x^\delta_{n,\alpha}) + \alpha I)^{-1} \int_0^1 [F'(x^\delta_{n,\alpha}) - F'(x^\delta_\alpha + t(x^\delta_{n,\alpha} - x^\delta_\alpha))]dt \\
&\quad (x^\delta_{n,\alpha} - x^\delta_\alpha). \qquad (2.3.2)
\end{aligned}
$$

So by Assumption 2.1.5, we have

$$
\begin{aligned}
\|x^\delta_{n+1,\alpha} - x^\delta_\alpha\| &\leq \|(F'(x^\delta_{n,\alpha}) + \alpha I)^{-1} F'(x^\delta_{n,\alpha})\| \\
&\quad \| \int_0^1 \phi(x^\delta_\alpha + t(x^\delta_{n,\alpha} - x^\delta_\alpha), x^\delta_{n,\alpha}, (x^\delta_{n,\alpha} - x^\delta_\alpha))dt\| \\
&\leq \frac{K_0}{2}\|x^\delta_{n,\alpha} - x^\delta_\alpha\|^2. \qquad (2.3.3)
\end{aligned}
$$

This shows that $x^\delta_{n,\alpha}$ converges quadratically to x^δ_α. The estimate (2.3.1) follows from (2.3.3). □

2.4 CONVERGENCE ANALYSIS OF (2.1.2) USING ASSUMPTION 2.1.6

Throughout this section we assume that $r < \frac{1}{3k_0}, \delta < C_r\alpha_0$ and $\alpha \geq \alpha_0$ where $C_r := (\frac{1-3k_0 r}{1-k_0 r})r$.

To prove the convergence of $(x^\delta_{n,\alpha})$ defined in (2.1.2) using the weaker Assumption 2.1.6, instead of the strong Assumption 2.1.5, we introduce the parameter $\rho > 0$.

Let $\|x_0 - \hat{x}\| \leq \rho$ with

$$\rho \leq \frac{1}{2}\left[C_r - \frac{\delta}{\alpha_0}\right]. \qquad (2.4.1)$$

THEOREM 2.4.1. *Suppose Assumption 2.1.6 holds and $r < \frac{1}{3k_0}$. Let $x^\delta_{n+1,\alpha}$*

be as in (2.1.2). Then $x^\delta_{n,\alpha}$ converges linearly to x^δ_α and

$$\|x^\delta_{n+1,\alpha} - x^\delta_\alpha\| \leq q^{n+1} r \tag{2.4.2}$$

where $q = \frac{2k_0 r}{1-k_0 r} < 1$.

Proof. For $x \in B(x_0, r)$, let $A_x = F'(x)$ and $A_n = F'(x^\delta_{n,\alpha})$. Then by Assumption 2.1.6, we have

$$\begin{aligned} \|(A_0+\alpha I)^{-1}(A_x - A_0)\| &= \sup_{\|v\|\leq 1} \|(A_0+\alpha I)^{-1}(A_x - A_0)v\| \\ &= \sup_{\|v\|\leq 1} \|(A_0+\alpha I)^{-1}A_0\phi(x, x_0, v)\| \\ &\leq k_0 r < 1. \end{aligned}$$

So, $I + (A_0+\alpha I)^{-1}(A_x - A_0)$ is invertible,

$$\|I + (A_0+\alpha I)^{-1}(A_x - A_0)\| \leq \frac{1}{1-k_0 r} \tag{2.4.3}$$

and

$$(A_x+\alpha I)^{-1} = [I + (A_0+\alpha I)^{-1}(A_x - A_0)]^{-1}(A_0+\alpha I)^{-1}. \tag{2.4.4}$$

By (2.3.2), we have

$$\begin{aligned} x^\delta_{n+1,\alpha} - x^\delta_\alpha &= (A_n+\alpha I)^{-1} \times \\ & \int_0^1 [A_n - F'(x^\delta_\alpha + t(x^\delta_{n,\alpha} - x^\delta_\alpha))]dt(x^\delta_{n,\alpha} - x^\delta_\alpha). \end{aligned} \tag{2.4.5}$$

If $x^\delta_{n,\alpha} \in B(x_0, r)$, then by (2.4.4) and (2.4.5), we have

$$\begin{aligned} x^\delta_{n+1,\alpha} - x^\delta_\alpha &= (A_n+\alpha I)^{-1}\int_0^1 [A_n - A_0 + A_0 - F'(x^\delta_\alpha + t(x^\delta_{n,\alpha} - x^\delta_\alpha))]dt \\ & \times (x^\delta_{n,\alpha} - x^\delta_\alpha) \\ &= (A_n+\alpha I)^{-1}[A_0 \int_0^1 \phi(x^\delta_{n,\alpha}, x_0, x^\delta_{n,\alpha} - x^\delta_\alpha)dt \end{aligned}$$

$$
\begin{aligned}
& -A_0 \int_0^1 \phi(x_\alpha^\delta + t(x_{n,\alpha}^\delta - x_\alpha^\delta), x_0, x_{n,\alpha}^\delta - x_\alpha^\delta)]dt \\
= \;& [I + (A_0 + \alpha I)^{-1}(A_n - A_0)](A_0 + \alpha I)^{-1} A_0 \\
& \times \int_0^1 [\phi(x_{n,\alpha}^\delta, x_0, x_{n,\alpha}^\delta - x_\alpha^\delta) \\
& -\phi(x_\alpha^\delta + t(x_{n,\alpha}^\delta - x_\alpha^\delta), x_0, x_{n,\alpha}^\delta - x_\alpha^\delta)]dt.
\end{aligned}
$$

So by Assumption 2.1.6 and (2.4.3) we have

$$
\begin{aligned}
\|x_{n+1,\alpha}^\delta - x_\alpha^\delta\| &\leq \frac{2k_0 r}{1 - k_0 r} \|x_{n,\alpha}^\delta - x_\alpha^\delta\| \\
&= q\|x_{n,\alpha}^\delta - x_\alpha^\delta\| \leq q^{n+1}\|x_0 - x_\alpha^\delta\| \\
&\leq q^{n+1} r. \qquad (2.4.6)
\end{aligned}
$$

This proves (2.4.2). Now it remains to prove that $x_{n,\alpha}^\delta \in B(x_0, r)$ for all $n > 0$. Note that, by the definition of x_α^δ, we have

$$
F(x_\alpha^\delta) - F(\hat{x}) + \alpha(x_\alpha^\delta - \hat{x}) = y^\delta - y + \alpha(x_0 - \hat{x}). \qquad (2.4.7)
$$

Since F is monotone, by taking inner product with $x_\alpha^\delta - \hat{x}$ on both sides of the above equation one can prove that

$$
\|x_\alpha^\delta - \hat{x}\| \leq \frac{\delta}{\alpha} + \|x_0 - \hat{x}\|.
$$

Therefore by the triangle inequality

$$
\begin{aligned}
\|x_\alpha^\delta - x_0\| &\leq \|x_\alpha^\delta - \hat{x}\| + \|\hat{x} - x_0\| \\
&\leq 2\rho + \frac{\delta}{\alpha_0}. \qquad (2.4.8)
\end{aligned}
$$

So by (2.4.6) and (2.4.8) we have,

$$
\begin{aligned}
\|x_{k+1,\alpha}^\delta - x_0\| &\leq \|x_{k+1,\alpha}^\delta - x_\alpha^\delta\| + \|x_\alpha^\delta - x_0\| \\
&\leq (q^k + 1)\|x_\alpha^\delta - x_0\| < \frac{1}{1-q}\left(2\rho + \frac{\delta}{\alpha_0}\right) \leq r.
\end{aligned}
$$

Thus by induction $x_n \in B(x_0, r)$ for all $n > 0$. $\square$

2.5 ERROR BOUNDS UNDER SOURCE CONDITIONS

Combining the estimates in Theorem 2.2.2 and Theorem 2.3.1 we obtain the following Theorem.

THEOREM 2.5.1. *Let $x^\delta_{n,\alpha}$ be as in (2.1.2) and let the assumptions in Theorem 2.2.2 and 2.3.1 be satisfied. Then we have the following;*

$$\|x^\delta_{n,\alpha} - \hat{x}\| \leq (\frac{K_0}{2})^{2^n-1} e^{-\gamma_r 2^n} + \frac{1}{1-K_0 r}(\frac{\delta}{\alpha} + \varphi(\alpha)). \tag{2.5.1}$$

Let

$$n_\delta := min\{n : e^{-\gamma_r 2^n} \leq \frac{\delta}{\alpha}\} \tag{2.5.2}$$

and let

$$C_1 := max\{\left(\frac{K_0}{2}\right)^{2^n-1}, \frac{1}{1-K_0 r}\}. \tag{2.5.3}$$

THEOREM 2.5.2. *Let $x^\delta_{n,\alpha}$ be as in (2.1.2) and let the assumptions in Theorem 2.5.1 be satisfied. Let n_δ be as in (2.5.2) and C_1 be as in (2.5.3). Then*

$$\|x^\delta_{n_\delta,\alpha} - \hat{x}\| \leq \tilde{C_1}(\varphi(\alpha) + \frac{\delta}{\alpha}) \tag{2.5.4}$$

where $\tilde{C_1} = 2C_1$.

Similarly, combining the estimates in Theorem 2.2.2 and Theorem 2.4.1 we obtain the following Theorem.

THEOREM 2.5.3. *Let $x^\delta_{n,\alpha}$ be as in (2.1.2) and let the assumptions in Theorem 2.2.2 and 2.4.1 be satisfied. Then we have the following;*

$$\|x^\delta_{n,\alpha} - \hat{x}\| \leq q^n r + \frac{1}{1-K_0 r}(\frac{\delta}{\alpha} + \varphi(\alpha)). \tag{2.5.5}$$

Let

$$n_\delta := min\{n : q^n \leq \frac{\delta}{\alpha}\} \tag{2.5.6}$$

and let

$$C_2 := max\{r, \frac{1}{1-K_0 r}\}. \tag{2.5.7}$$

THEOREM 2.5.4. *Let $x^\delta_{n,\alpha}$ be as in (2.1.2) and let the assumptions in Theorem 2.5.3 be satisfied. Let n_δ be as in (2.5.6) and C_2 be as in (2.5.7). Then*

$$\|x^\delta_{n_\delta,\alpha} - \hat{x}\| \le \tilde{C}_2(\varphi(\alpha) + \frac{\delta}{\alpha}) \tag{2.5.8}$$

where $\tilde{C}_2 = 2C_2$.

2.5.1 A priori choice of the parameter

Note that the error $\varphi(\alpha) + \frac{\delta}{\alpha}$ in (2.5.4) or (2.5.8) is of optimal order if $\alpha_\delta := \alpha(\delta)$ satisfies, $\alpha_\delta\varphi(\alpha_\delta) = \delta$. Now using the function $\psi(\lambda) := \lambda\varphi^{-1}(\lambda), 0 < \lambda \le a$, we have $\delta = \alpha_\delta\varphi(\alpha_\delta) = \psi(\varphi(\alpha_\delta))$, so that $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$. Hence by (2.5.4) or (2.5.8) we have the following.

THEOREM 2.5.5. *Let $\psi(\lambda) := \lambda\varphi^{-1}(\lambda)$ for $0 < \lambda \le a$, and let the assumptions in Theorem 2.5.2 or Theorem 2.5.4 holds. For $\delta > 0$, let $\alpha := \alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$. Let n_δ be as in (2.5.2) or (2.5.6). Then*

$$\|x^\delta_{n_\delta,\alpha} - \hat{x}\| = O(\psi^{-1}(\delta)).$$

The regularization parameter α is selected from some finite set

$$D_M(\alpha) := \{\alpha_i = \mu^i\alpha_0, i = 0, 1, \cdots, M\},$$

where $\mu > 1, \alpha_0 > 0$.

Let

$$n_i := min\{n : e^{-\gamma_r 2^n} \le \frac{\delta}{\alpha_i}\}$$

or

$$n_i := min\{n : q^n \le \frac{\delta}{\alpha_i}\}.$$

Then for $i = 0, 1, \cdots, M$, we have

$$\|x^\delta_{n_i,\alpha_i} - x^\delta_{\alpha_i}\| \le (\frac{K_0}{2})^{2^{n_i-1}}\frac{\delta}{\alpha_i}, \quad \forall i = 0, 1, \cdots, M$$

or

$$\|x^\delta_{n_i,\alpha_i} - x^\delta_{\alpha_i}\| \le r\frac{\delta}{\alpha_i}, \quad \forall i = 0, 1, \cdots, M.$$

Let $x_i := x^\delta_{n_i,\alpha_i}$. The parameter choice strategy that we are going to consider here, we select $\alpha = \alpha_i$ from $D_M(\alpha)$ and operates only with corresponding $x_i, i = 0, 1, \cdots, M$. The proof of the following Theorem is analogous to the proof of Theorem 4.4 in George (2010)(see also George and Nair (2008)) so we ignore the details.

THEOREM 2.5.6. *Assume that there exists $i \in \{0, 1, 2, \cdots, M\}$ such that $\varphi(\alpha_i) \le \frac{\delta}{\alpha_i}$. Let assumptions of Theorem 2.5.1 and Theorem 2.5.2 hold and let*

$$l := max\{i : \varphi(\alpha_i) \le \frac{\delta}{\alpha_i}\} < M,$$

$$k := max\{i : \|x_i - x_j\| \le 4\tilde{C}\frac{\delta}{\alpha_j}, j = 0, 1, 2, \cdots, i-1\}.$$

Then $l \le k$ and

$$\|\hat{x} - x_k\| \le c\psi^{-1}(\delta),$$

where $c = 6\tilde{C}\mu$.

2.6 CONCLUSION

In this chapter we considered an iterative method, it can be considered as a modified iteratively regularized Lavrentiev method for solving nonlinear ill-posed operator equation (1.2.1). Note that, in the iteratively regularized Lavrentiev method for approximately solving the nonlinear ill-posed equation (1.2.1) with a monotone operator F involves finding a fixed point of $G(x) := x - (F'(x) + \alpha I)^{-1}[F(x) - y^\delta + \alpha(x - x_0)]$, where x_0 is an initial guess which may incorporate a priori knowledge of an exact solution and y^δ. Precisely we considered the sequence $(x^\delta_{n,\alpha})$ defined iteratively by $x^\delta_{n+1,\alpha} = G(x^\delta_{n,\alpha})$ for obtaining the fixed point of $G(x)$.

In the next chapter we consider the finite dimensional realization and the implementation of the method considered in this chapter.

Chapter 3

FINITE DIMENSIONAL REALIZATION OF A QUADRATIC CONVERGENCE YIELDING ITERATIVE REGULARIZATION METHOD FOR ILL-POSED EQUATIONS WITH MONOTONE OPERATORS

In Chapter 2 we considered a quadratic convergent iterative method for obtaining approximate solution to nonlinear ill posed operator equation $F(x) = y$, where $F : D(F) \subseteq X \to X$ is a monotone operator and X real Hilbert space. In this chapter we consider the finite dimensional realization of the method considered in Chapter 2. A numerical example is given to justify our theoretical results.

3.1 INTRODUCTION

Let $\{P_h\}_{h>0}$ be a family of orthogonal projections on X onto $R(P_h)$, the range of P_h. Our aim is to obtain an approximation for x_α^δ, in the finite dimensional space $R(P_h)$. For the results that follow, we impose the following conditions.

Let

$$\epsilon_h := \|F'(.)(I - P_h)\|,$$

and

$$b_h := \|(I - P_h)\hat{x}\|.$$

We assume that $\lim_{h\longrightarrow 0} \epsilon_h = 0$ and $\lim_{h\to 0} b_h = 0$. The above assumption is satisfied if $P_h \to I$ point wise and if $F'(.)$ is compact operator. Further we assume that there exist $\epsilon_0 > 0$, $b_0 > 0$ and $\delta_0 > 0$ such that $\epsilon_h < \epsilon_0, b_h < b_0$ and $\delta < \delta_0$.

3.2 THE PROJECTION METHOD AND ITS CONVERGENCE

We consider the iterative method defined for $n = 0, 1, 2, \cdots$ by

$$x_{n+1,\alpha}^{h,\delta} = x_{n,\alpha}^{h,\delta} - R_\alpha^{-1}(x_{n,\alpha}^{h,\delta})P_h[F(x_{n,\alpha}^{h,\delta}) - y^\delta + \alpha(x_{n,\alpha}^{h,\delta} - x_0)], \qquad (3.2.1)$$

where $R_\alpha(x) := P_h F'(x) P_h + \alpha P_h$ and $x_{0,\alpha}^{h,\delta} := P_h x_0$ as an approximation for $x_\alpha^{h,\delta}$. First we prove that

$$P_h F P_h(x) + \alpha P_h(x - x_0) = P_h y^\delta \qquad (3.2.2)$$

has a unique solution $x_\alpha^{h,\delta}$ in $B(P_h x_0, r)$ and then we prove that the sequence $(x_{n,\alpha}^{h,\delta})$ defined in (3.2.1) converges quadratically to $x_\alpha^{h,\delta}$.

Let

$$r \geq 2(r_0 + \max\{1, \|\hat{x}\|\}) \text{ with } r_0 := \|\hat{x} - x_0\|.$$

Here after we assume that $\epsilon_h \in (0, \epsilon_0), \delta \in [0, \delta_0]$, $a_0 \geq \varepsilon_0 + \delta_0$, $\alpha \in (\delta + \epsilon_h, a_0)$.

PROPOSITION 3.2.1. *Let F be a monotone operator and $\{P_h\}_{h>0}$ be a family of orthogonal projections of X onto $R(P_h)$ with $R(P_h) \subset D(F)$. Then $P_h F P_h$ is a monotone operator on X and the operator equation (3.2.2) has a unique solution $x_\alpha^{h,\delta}$ for all $x_0, y^\delta \in X$. Furthermore $x_\alpha^{h,\delta} \in B(P_h x_0, r)$.*

Proof. Since F is monotone and $R(P_h) \subset D(F)$, we have

$$\langle P_hFP_h(x) - P_hFP_h(y), x - y\rangle = \langle F(P_hx) - F(P_hy), P_hx - P_hy\rangle \geq 0.$$

That is P_hFP_h is monotone operator and $D(P_hFP_h) = X$. Therefore by Browder–Minty theorem (Alber and Ryazantseva (2006), Theorem 1.15.21) $R(P_hFP_h+\alpha I) = X$ (see also, Pereverzev and Schock (2005)) and $P_hFP_h+\alpha I$ is injective. Hence the equation

$$(P_hFP_h + \alpha I)x = P_h(y^\delta + \alpha x_0)$$

has a unique solution $x_\alpha^{h,\delta}$. Note that $x_\alpha^{h,\delta}$ satisfies

$$P_hF(x_\alpha^{h,\delta}) + \alpha P_h(x_\alpha^{h,\delta} - x_0) = P_hy^\delta. \tag{3.2.3}$$

Let $M_h = \int_0^1 F'(\hat{x} + t(x_\alpha^{h,\delta} - \hat{x}))dt$. Then, by (3.2.3), we have

$$(P_hM_hP_h + \alpha I)(x_\alpha^{h,\delta} - P_h\hat{x}) = \alpha P_h(x_0 - \hat{x}) + P_h(y^\delta - y) + (P_hM_h(I - P_h))(\hat{x})$$

and hence

$$\begin{aligned}
\|x_\alpha^{h,\delta} - P_h\hat{x}\| &\leq \|(P_hM_hP_h + \alpha I)^{-1} \\
&\quad \times[\alpha P_h(x_0 - \hat{x}) + P_h(y^\delta - y) \\
&\quad +(P_hM_h(I - P_h))(\hat{x})]\| \\
&\leq \|P_h(x_0 - \hat{x})\| + \frac{\|P_h(y^\delta - y)\|}{\alpha} \\
&\quad +\frac{\|P_hM_h(I - P_h)\|\|\hat{x}\|}{\alpha} \\
&\leq r_0 + \frac{\delta}{\alpha} + \frac{\varepsilon_h\|\hat{x}\|}{\alpha}.
\end{aligned} \tag{3.2.4}$$

Therefore,

$$\|x_\alpha^{h,\delta} - P_hx_0\| \leq \|x_\alpha^{h,\delta} - P_h\hat{x}\| + \|P_h(\hat{x} - x_0)\|$$

$$
\begin{aligned}
&\leq 2r_0 + \frac{\delta}{\alpha} + \frac{\varepsilon_h\|\hat{x}\|}{\alpha} \\
&\leq 2r_0 + \max\{1, \|\hat{x}\|\}\frac{\varepsilon_h + \delta}{\alpha} \\
&\leq 2r_0 + \max\{1, \|\hat{x}\|\} < r,
\end{aligned}
$$

that is $x_\alpha^{h,\delta} \in B(P_h x_0, r)$. □

For convenience we define $e_n := \|x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}\|$. We need the following Lemma for proving our results.

LEMMA 3.2.2. *Let $x \in D(F)$. Then $\|(P_hF'(x)P_h + \alpha I)^{-1}P_hF'(x)\| \leq 2$.*

Proof. Observe that,

$$
\begin{aligned}
\|(P_hF'(x)P_h + \alpha I)^{-1}P_hF'(x)\| &= \sup_{\|v\|\leq 1} \|(P_hF'(x)P_h + \alpha I)^{-1}P_hF'(x)v\| \\
&= \sup_{\|v\|\leq 1} \|(P_hF'(x)P_h + \alpha I)^{-1} \\
&\quad P_hF'(x)(P_h + I - P_h)v\| \\
&\leq \sup_{\|v\|\leq 1} \|(P_hF'(x)P_h + \alpha I)^{-1}P_hF'(x)(P_h)v\| \\
&\quad + \sup_{\|v\|\leq 1} \|(P_hF'(x)P_h + \alpha I)^{-1}P_hF'(x)(I - P_h)v\| \\
&\leq 1 + \frac{\epsilon_h}{\alpha} \leq 2.
\end{aligned}
$$

□

THEOREM 3.2.3. *Let Assumption 2.1.5 holds and $x_{n+1,\alpha}^{h,\delta}$ be as in (3.2.1). Then $x_{n,\alpha}^{h,\delta}$ converges quadratically to $x_\alpha^{h,\delta}$ and*

$$
\|x_{n+1,\alpha}^{h,\delta} - x_\alpha^{h,\delta}\| \leq K_0^{2^{n+1}-1} r^{2^{n+1}}. \tag{3.2.5}
$$

Proof. Using (3.2.3) and Assumption 2.1.5, we have

$$\begin{aligned}
x_{n+1,\alpha}^{h,\delta} - x_\alpha^{h,\delta} &= x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta} - (P_h F'(x_{n,\alpha}^{h,\delta})P_h + \alpha I)^{-1} \\
&\quad \times P_h[F(x_{n,\alpha}^{h,\delta}) - F(x_\alpha^{h,\delta}) + \alpha(x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta})] \\
&= (P_h F'(x_{n,\alpha}^{h,\delta})P_h + \alpha I)^{-1}[(P_h F'(x_{n,\alpha}^{h,\delta})P_h + \alpha I)(x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}) \\
&\quad - P_h(F(x_{n,\alpha}^{h,\delta}) - F(x_\alpha^{h,\delta})) - \alpha(x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta})] \\
&= (P_h F'(x_{n,\alpha}^{h,\delta})P_h + \alpha I)^{-1}[P_h F'(x_{n,\alpha}^{h,\delta})(x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}) \\
&\quad - P_h(F(x_{n,\alpha}^{h,\delta}) - F(x_\alpha^{h,\delta}))] \\
&= (P_h F'(x_{n,\alpha}^{h,\delta})P_h + \alpha I)^{-1} P_h[F'(x_{n,\alpha}^{h,\delta})(x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}) \\
&\quad - (F(x_{n,\alpha}^{h,\delta}) - F(x_\alpha^{h,\delta}))] \\
&= (P_h F'(x_{n,\alpha}^{h,\delta})P_h + \alpha I)^{-1} \\
&\quad \times P_h[F'(x_{n,\alpha}^{h,\delta}) - \int_0^1 F'(x_\alpha^{h,\delta} + t(x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}))dt](x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}) \\
&= (P_h F'(x_{n,\alpha}^{h,\delta})P_h + \alpha I)^{-1} P_h[\int_0^1 (F'(x_{n,\alpha}^{h,\delta}) \\
&\quad - F'(x_\alpha^{h,\delta} + t(x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta})))dt](x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}) \qquad (3.2.6) \\
&= -(P_h F'(x_{n,\alpha}^{h,\delta})P_h + \alpha I)^{-1} P_h F'(x_{n,\alpha}^{h,\delta}) \\
&\quad P_h \int_0^1 \phi(x_\alpha^{h,\delta} + t(x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}), x_{n,\alpha}^{h,\delta}, x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta})dt. \qquad (3.2.7)
\end{aligned}$$

Therefore, by (3.2.7), Lemma 3.2.2 and Assumption 2.1.5, we have in turn

$$\|x_{n+1,\alpha}^{h,\delta} - x_\alpha^{h,\delta}\| \leq 2\frac{K_0}{2}\|x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}\|^2 \leq K_0 e_n^2, \qquad (3.2.8)$$

i.e., $x_{n,\alpha}^{h,\delta}$ converges quadratically to $x_\alpha^{h,\delta}$. The estimate (3.2.5) follows from (3.2.8) and the fact that $e_0 \leq r$. □

THEOREM 3.2.4. *Let Assumption 2.1.6 holds and $x_{n+1,\alpha}^{h,\delta}$ be as in (3.2.1). Let $0 < r < \frac{1}{4K_0}$. Then $x_{n,\alpha}^{h,\delta}$ converges linearly to $x_\alpha^{h,\delta}$,*

$$\|x_{n+1,\alpha}^{h,\delta} - x_\alpha^{h,\delta}\| \leq q^{n+1} r, \qquad (3.2.9)$$

where $q = \frac{2K_0 r}{1-2K_0 r}$.

Proof. For $x \in B_{2r}(P_h x_0) \cap R(P_h)$ let $A_x = F'(x)$ and $A_0 = F'(P_h x_0)$. Then by Assumption 2.1.6 we have

$$\begin{aligned}
& \|(P_h A_0 P_h + \alpha I)^{-1} P_h (A_x - A_0) P_h\| \\
= & \sup_{\|v\| \leq 1} \|(P_h A_0 P_h + \alpha I)^{-1} P_h (A_x - A_0) P_h v\| \\
= & \sup_{\|v\| \leq 1} \|(P_h A_0 P_h + \alpha I)^{-1} P_h A_0 \phi(x, P_h x_0, P_h v)\| \\
= & \sup_{\|v\| \leq 1} \|(P_h A_0 P_h + \alpha I)^{-1} P_h A_0 [P_h + I - P_h] \phi(x, P_h x_0, P_h v)\| \\
\leq & k_0 [1 + \frac{\varepsilon_h}{\alpha}] r \leq 2 k_0 r < 1.
\end{aligned}$$

So, $I + (P_h A_0 P_h + \alpha I)^{-1} P_h (A_x - A_0) P_h$ is invertible,

$$\|I + (P_h A_0 P_h + \alpha I)^{-1} P_h (A_x - A_0) P_h\| \leq \frac{1}{1 - 2k_0 r} \tag{3.2.10}$$

and

$$\begin{aligned}
(P_h A_x P_h + \alpha I)^{-1} = & [I + (P_h A_0 P_h + \alpha I)^{-1} P_h (A_x - A_0) P_h]^{-1} \\
& \times (P_h A_0 P_h + \alpha I)^{-1}. \qquad (3.2.11)
\end{aligned}$$

By (3.2.6), we have

$$\begin{aligned}
x_{n+1,\alpha}^{h,\delta} - x_\alpha^{h,\delta} = & (P_h A_n P_h + \alpha I)^{-1} P_h \int_0^1 [A_n - F'(x_{n,\alpha}^\delta + t(x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}))] dt \\
& \times (x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}), \qquad (3.2.12)
\end{aligned}$$

where, here and below $A_n = F'(x_{n,\alpha}^{h,\delta})$. Suppose $x_{n,\alpha}^{h,\delta} \in B_r(P_h x_0)$, then by (3.2.11) and (3.2.12), we have

$$\begin{aligned}
& x_{n+1,\alpha}^{h,\delta} - x_\alpha^{h,\delta} \\
= & (P_h A_n P_h + \alpha I)^{-1} \int_0^1 P_h [A_n - A_0 + A_0 - F'(x_\alpha^{h,\delta} + t(x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}))] P_h dt \\
& \times (x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta})
\end{aligned}$$

$$
\begin{aligned}
&= (P_hA_nP_h+\alpha I)^{-1}P_h[A_0\int_0^1 \phi(x^{\delta}_{n,\alpha}, P_hx_0, x^{h,\delta}_{n,\alpha}-x^{h,\delta}_{\alpha})dt\\
&\quad -P_hA_0\int_0^1 \phi(x^{h,\delta}_{\alpha}+t(x^{h,\delta}_{n,\alpha}-x^{h,\delta}_{\alpha}), P_hx_0, x^{h,\delta}_{n,\alpha}-x^{h,\delta}_{\alpha})]dt\\
&= [I+(P_hA_0P_h+\alpha I)^{-1}P_h(A_n-A_0)P_h]^{-1}(P_hA_0P_h+\alpha I)^{-1}P_hA_0\\
&\quad \times[P_h+I-P_h]\int_0^1 [\phi(x^{h,\delta}_{n,\alpha}, P_hx_0, x^{h,\delta}_{n,\alpha}-x^{h,\delta}_{\alpha})\\
&\quad -\phi(x^{h,\delta}_{\alpha}+t(x^{h,\delta}_{n,\alpha}-x^{h,\delta}_{\alpha}), P_hx_0, x^{h,\delta}_{n,\alpha}-x^{h,\delta}_{\alpha})]dt.
\end{aligned}
$$

So by Assumption 2.1.6 and (3.2.10) we have

$$
\begin{aligned}
\|x^{h,\delta}_{n+1,\alpha}-x^{h,\delta}_{\alpha}\| &\leq \frac{1}{1-2k_0r}[2k_0r]\,\|x^{h,\delta}_{n,\alpha}-x^{h,\delta}_{\alpha}\|\\
&\leq q\|x^{h,\delta}_{n,\alpha}-x^{h,\delta}_{\alpha}\| \leq q^n\|P_hx_0-x^{h,\delta}_{\alpha}\|\\
&\leq q^n r. \qquad (3.2.13)
\end{aligned}
$$

Therefore $(x^{h,\delta}_{n,\alpha})$ converges to $x^{h,\delta}_{\alpha}$ as $n\to\infty$. Now it remains to prove that $x^{h,\delta}_{n,\alpha}\in B_{2r}(x_0)$ for all $n>0$. Note that $0<q<1$ and hence

$$
\begin{aligned}
\|x^{h,\delta}_{n,\alpha}-P_hx_0\| &\leq q\|x^{h,\delta}_{n-1,\alpha}-x^{h,\delta}_{\alpha}\|+\|x^{h,\delta}_{\alpha}-P_hx_0\|\\
&\leq q^n\|x^{h,\delta}_{\alpha}-P_hx_0\|+\|x^{h,\delta}_{\alpha}-P_hx_0\|\\
&\leq 2r.
\end{aligned}
$$

Therefore, since $x^{h,\delta}_{\alpha}\in B_r(P_hx_0)\subset B_{2r}(P_hx_0)$, by using an induction argument one can prove that $x^{h,\delta}_{n,\alpha}\in B_{2r}(P_hx_0)$ for all $n>0$. □

REMARK 3.2.5. *1. As mentioned in Chapter 2, the applicability of (3.2.1) is extended to a large domain under the weaker Assumption 2.1.6 because the convergence domain of (3.2.1) with Assumption 2.1.6 can be large, since $\frac{K_0}{k_0}$ can be arbitrarily small.(see for e.g., Example 7.3 in Argyros et al. (2013)).*

2. Instead of Assumption 2.1.5, if we use the following Lipschitz condition:

$$\|F'(x_1)-F'(x_2)\|\leq L\|x_1-x_2\| \qquad (3.2.14)$$

then from (3.2.6) we have

$$\begin{aligned} \|x_{n+1,\alpha}^{h,\delta} - x_\alpha^{h,\delta}\| &\leq \|P_h F'(x_{n,\alpha}^{h,\delta}) P_h + \alpha I)^{-1}\| \\ &\quad \times \| \int_0^1 [(F'(x_\alpha^{h,\delta} + t(x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta})) - F'(x_{n,\alpha}^{h,\delta}))dt] \| \\ &\quad \times \|x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}\| \\ &\leq \frac{1}{\alpha}\frac{L}{2}\|x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}\|^2 \qquad (3.2.15) \\ &\leq \left(\frac{L}{2\alpha}\right)^{2^{n+1}-1} \|x_{0,\alpha}^{h,\delta} - x_\alpha^{h,\delta}\|^{2^{n+1}} \\ &= \left(\frac{Le_0}{2\alpha}\right)^{2^{n+1}-1} e_0 \\ &\leq \left(\frac{Lr}{2\alpha}\right)^{2^{n+1}-1} r. \qquad (3.2.16) \end{aligned}$$

Thus if $L < 2\alpha$, *(or* $\frac{Lr}{2\alpha} < 1$,*) then by (3.2.15)(or (3.2.16))* $x_{n+1,\alpha}^{h,\delta}$ *converges to* $x_\alpha^{h,\delta}$.

3.3 ERROR BOUNDS UNDER SOURCE CONDITIONS

In this section we obtain an error estimate for $\|x_{n,\alpha}^{h,\delta} - \hat{x}\|$ under the source condition (2.1.3). As in Proposition 3.2.1, one can prove that

$$P_h F P_h(x) + \alpha P_h(x - x_0) = P_h y \qquad (3.3.1)$$

has a unique solution $x_\alpha^h \in B(P_h x_0, r)$.

PROPOSITION 3.3.1. *Let* $F : D(F) \subseteq X \to X$ *be a monotone operator in* X*. Let* $x_\alpha^{h,\delta}$ *be the solution of (3.2.3) and* x_α^h *be the unique solution of (3.3.1). Then*

$$\|x_\alpha^{h,\delta} - x_\alpha^h\| \leq \frac{\delta}{\alpha}. \qquad (3.3.2)$$

Proof. Let $F_h = \int_0^1 F'(x_\alpha^h + t(x_\alpha^{h,\delta} - x_\alpha^h))dt$. Then, by (3.2.3) and (3.3.1), we have

$$P_h[F(x_\alpha^{h,\delta}) - F(x_\alpha^h)] + \alpha(x_\alpha^{h,\delta} - x_\alpha^h) = P_h(y^\delta - y)$$

and hence

$$(P_hF_hP_h + \alpha I)(x_\alpha^{h,\delta} - x_\alpha^h) = P_h(y^\delta - y).$$

So

$$\|x_\alpha^{h,\delta} - x_\alpha^h\| \leq \|(P_hF_hP_h + \alpha I)^{-1}P_h(y^\delta - y)\| \leq \frac{\delta}{\alpha}.$$

This completes the proof. □

THEOREM 3.3.2. *Let Assumption 2.1.4 and Assumption 2.1.5 hold. Let $\hat{x}$ be the solution of (1.2.1), $0 < r < \frac{1}{K_0}$ and x_α^h be the solution of (3.3.1). Then*

$$\|x_\alpha^h - \hat{x}\| \leq \tilde{C}[\varphi(\alpha) + \frac{\epsilon_h}{\alpha} + b_h] \tag{3.3.3}$$

where $\tilde{C} = \frac{\max\{2(K_0r_0+1), \|\hat{x}\|\}}{1-K_0r_0}$.

Proof. Let $M_\alpha := \int_0^1 F'(\hat{x} + t(x_\alpha^h - \hat{x}))dt$. Then since x_α^h satisfies

$$P_hF(x_\alpha^h) + \alpha P_h(x_\alpha^h - x_0) = P_hy,$$

we have $P_hM_\alpha(x_\alpha^h - \hat{x}) + \alpha P_h(x_\alpha^h - x_0) = 0$. Hence,

$$(P_hM_\alpha P_h + \alpha I)(x_\alpha^h - P_h\hat{x}) = \alpha P_h(x_0 - \hat{x}) + P_hM_\alpha(I - P_h)\hat{x}.$$

Therefore,

$$\begin{aligned}
x_\alpha^h - P_h\hat{x} &= [(P_hM_\alpha P_h + \alpha I)^{-1}P_h - P_h(F'(x_0) + \alpha I)^{-1}]\alpha(x_0 - \hat{x}) \\
&\quad + P_h(F'(x_0) + \alpha I)^{-1}\alpha(x_0 - \hat{x}) \\
&\quad + (P_hM_\alpha P_h + \alpha I)^{-1}P_hM_\alpha(I - P_h)\hat{x} \\
&= (P_hM_\alpha P_h + \alpha I)^{-1}P_h[F'(x_0) - M_\alpha + M_\alpha(I - P_h)] \\
&\quad (F'(x_0) + \alpha I)^{-1}\alpha(x_0 - \hat{x}) \\
&\quad + P_h(F'(x_0) + \alpha I)^{-1}\alpha(x_0 - \hat{x}) + (P_hM_\alpha P_h + \alpha P_h)^{-1}P_hM_\alpha(I - P_h)\hat{x} \\
&:= \Gamma_1 + \Gamma_2,
\end{aligned} \tag{3.3.4}$$

where

$$\Gamma_1 := (P_hM_\alpha P_h + \alpha I)^{-1}P_h[F'(x_0) - M_\alpha + M_\alpha(I - P_h)](F'(x_0) + \alpha I)^{-1}\alpha(x_0 - \hat{x})$$

and

$$\Gamma_2 := P_h(F'(x_0)+\alpha I)^{-1}\alpha(x_0-\hat{x}) + (P_hM_\alpha P_h+\alpha P_h)^{-1}P_hM_\alpha(I-P_h)\hat{x}.$$

By Assumption 2.1.4, 2.1.5 and Lemma 3.2.2, we have

$$\begin{aligned}
\|\Gamma_1\| &\leq \|(P_hM_\alpha P_h+\alpha I)^{-1}P_h\int_0^1[F'(\hat{x}+t(x_\alpha^h-\hat{x}))\\
&\quad \phi(x_0,\hat{x}+t(x_\alpha^h-\hat{x}),(F'(x_0)+\alpha I)^{-1}\alpha(x_0-\hat{x}))]dt\|\\
&\quad +\frac{\epsilon_h}{\alpha}\|(F'(x_0)+\alpha I)^{-1}\alpha(x_0-\hat{x})\|\\
&\leq \|(P_hM_\alpha P_h+\alpha I)^{-1}P_hM_\alpha\|\|\phi(x_0,\hat{x}+t(x_\alpha^h-\hat{x}),(F'(x_0)+\alpha I)^{-1}\alpha(x_0-\hat{x}))dt\|\\
&\quad +\frac{\epsilon_h}{\alpha}\|(F'(x_0)+\alpha I)^{-1}\alpha(x_0-\hat{x})\|\\
&\leq 2K_0[\|x_0-\hat{x}\|+\frac{1}{2}\|x_\alpha^h-\hat{x}\|]\|(F'(x_0)+\alpha I)^{-1}\alpha(x_0-\hat{x})\|+\frac{\epsilon_h}{\alpha}\varphi(\alpha)\\
&\leq 2K_0\|x_0-\hat{x}\|\varphi(\alpha)+2\frac{K_0}{2}\|x_\alpha^h-\hat{x}\|\|x_0-\hat{x}\|+\frac{\epsilon_h}{\alpha}\varphi(\alpha)\\
&\leq 2K_0r_0\varphi(\alpha)+\varphi(\alpha)+K_0r_0\|x_\alpha^h-\hat{x}\|\\
&\leq (2K_0r_0+1)\varphi(\alpha)+K_0r_0\|x_\alpha^h-\hat{x}\| \qquad (3.3.5)
\end{aligned}$$

and

$$\|\Gamma_2\| \leq \varphi(\alpha)+\frac{\epsilon_h}{\alpha}\|\hat{x}\|. \qquad (3.3.6)$$

Now the result (3.3.3) follows from (3.3.4), (3.3.5) and (3.3.6) and the following triangle inequality

$$\|x_\alpha^h-\hat{x}\| \leq \|x_\alpha^h-P_h\hat{x}\|+\|(I-P_h)\hat{x}\|.$$

□

THEOREM 3.3.3. *Let the assumptions in Theorem 3.2.3 and Theorem 3.3.2 be satisfied. Let $x_{n,\alpha}^{h,\delta}$ be as in (3.2.1) and $0<r<\frac{1}{k_0}$. Then*

$$\|x_{n,\alpha}^{h,\delta}-\hat{x}\| \leq (K_0r)^{2^n-1}r+\max\{1,\tilde{C}\}[\varphi(\alpha)+\frac{\delta+\epsilon_h}{\alpha}+b_h]. \qquad (3.3.7)$$

Proof. Note that,

$$\|x_{n,\alpha}^{h,\delta} - \hat{x}\| \leq \|x_{n,\alpha}^{h,\delta} - x_{\alpha}^{h,\delta}\| + \|x_{\alpha}^{h,\delta} - x_{\alpha}^{h}\| + \|x_{\alpha}^{h} - \hat{x}\|.$$

So by Proposition 3.3.1, Theorem 3.2.3 and Theorem 3.3.2 we have,

$$\begin{aligned} \|x_{n,\alpha}^{h,\delta} - \hat{x}\| &\leq (K_0 r)^{2^n - 1} r + \frac{\delta}{\alpha} + \tilde{C}(\varphi(\alpha) + \frac{\epsilon_h}{\alpha} + b_h) \\ &\leq (K_0 r)^{2^n - 1} r + \max\{1, \tilde{C}\}[\varphi(\alpha) + \frac{\delta + \epsilon_h}{\alpha} + b_h]. \end{aligned}$$

□

Let $b_h \leq \frac{\delta+\epsilon_h}{\alpha}$,

$$n_\delta := \min\{n : (K_0 r)^{2^n - 1} \leq \frac{\delta + \epsilon_h}{\alpha}\} \tag{3.3.8}$$

and

$$C_0 = r + 2\max\{1, \tilde{C}\}. \tag{3.3.9}$$

THEOREM 3.3.4. *Let n_δ and C_0 be as in (3.3.8) and (3.3.9), respectively. Let $x_{n_\delta,\alpha}^{h,\delta}$ be as in (3.2.1) and assumption in Theorem 3.3.3 be satisfied. Then*

$$\|x_{n_\delta,\alpha}^{h,\delta} - \hat{x}\| \leq C_0(\varphi(\alpha) + \frac{\delta + \epsilon_h}{\alpha}). \tag{3.3.10}$$

3.3.1 A priori choice of the parameter

Note that the error estimate $\varphi(\alpha) + \frac{\delta+\epsilon_h}{\alpha}$ in (3.3.10) is of optimal order if $\alpha_\delta := \alpha(\delta, h)$ satisfies, $\varphi(\alpha_\delta)\alpha_\delta = \delta + \epsilon_h$.

Now using the function $\psi(\lambda) := \lambda\varphi^{-1}(\lambda), 0 < \lambda \leq a$, we have $\delta + \epsilon_h = \alpha_\delta\varphi(\alpha_\delta) = \psi(\varphi(\alpha_\delta))$, i.e., $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta + \epsilon_h))$. In view of the above observations and (3.3.10) we have the following theorem.

THEOREM 3.3.5. *Let $\psi(\lambda) := \lambda\varphi^{-1}(\lambda)$ for $0 < \lambda \leq a$, and the assumptions in Theorem 3.3.4 hold. For $\delta > 0$, let $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta + \epsilon_h)), b_h \leq \frac{\delta+\epsilon_h}{\alpha}$ and n_δ be as in (3.3.8). Then*

$$\|x_{n_\delta,\alpha}^{h,\delta} - \hat{x}\| = O(\psi^{-1}(\delta + \epsilon_h)).$$

3.3.2 An adaptive choice of the parameter

As in section 2.5, we consider the balancing principle for choosing the parameter α. Precisely, the regularization parameter α is selected from finite set

$$D_N(\alpha) := \{\alpha_i = \mu^i \alpha_0, i = 0, 1, 2, ..., N\}$$

where $\mu > 1, \alpha_0 > 0$. Let

$$n_i := \min\{n : (K_0 r)^{2^n - 1} \leq \frac{\delta + \epsilon_h}{\alpha_i}\}.$$

Let $x_i^h := x_{n_i,\alpha_i}^{h,\delta}$. We select $\alpha = \alpha_i$ from $D_N(\alpha)$ for computing x_i, for each $i = 0, 1, 2, ..., N$.

THEOREM 3.3.6. *Assume that there exists* $i \in \{0, 1, 2, ..., N\}$ *such that* $\varphi(\alpha_i) \leq \frac{\delta+\epsilon_h}{\alpha_i}$. *Let the assumptions of Theorem 3.3.4 and Theorem 3.3.5 be satisfied and let*

$$l := \max\{i : \varphi(\alpha_i) \leq \frac{\delta + \epsilon_h}{\alpha_i}\} < N,$$

$$k := \max\{i : \|x_i^h - x_j^h\| \leq 4C_0 \frac{\delta + \epsilon_h}{\alpha_j}, j = 0, 1, 2, ..., i\}.$$

Then $l \leq k$ *and* $\|\hat{x} - x_k^h\| \leq c\psi^{-1}(\delta + \epsilon_h)$ *where* $c = 6C_0\mu$.

3.4 IMPLEMENTATION OF THE ADAPTIVE CHOICE RULE

Finally the adaptive algorithm associated with the choice of the parameter specified in Theorem 3.3.6 involves the following steps:

- Choose $\alpha_0 = \delta + \epsilon_h$.
- $\alpha_i = \mu^i \alpha_0, i = 0, 1, 2, \cdots, M$.

3.4.1 Algorithm

1. Set $i \leftarrow 0$.

2. Solve $x_i := x^{\delta}_{n_i,\alpha_i}$ by using the iteration (3.2.1).

3. If $\|x_i - x_j\| > 4\tilde{C}\frac{1}{\mu^j}, \quad j < i$, then take $k = i$.

4. Set $i = i + 1$ and return to step 2.

3.5 IMPLEMENTATION OF THE METHOD

Let X_M be a sequence of finite dimensional subspaces of X and let $P_h, (h = \frac{1}{M})$ denote the orthogonal projection on X with $R(P_h) = X_M$. Let $\{\Phi_1, \Phi_2, \ldots, \Phi_M\}$ be a basis for X_M. We assume that $\|P_h x - x\| \to 0$ as $h \to 0$ $\forall x \in X$.

Since $x^{h,\delta}_{n,\alpha} \in X_M$, there exist $\lambda^n_1, \lambda^n_2, ..., \lambda^n_M \in \mathbb{R}$, such that $x^{h,\delta}_{n,\alpha} = \sum_{i=1}^{M} \lambda^n_i \Phi_i$. Then from (3.2.1) we have,

$$\begin{aligned}
&(P_h F'(x^{h,\delta}_{n,\alpha}) + \alpha I)\sum_{i=1}^{M}(\lambda^{n+1}_i - \lambda^n_i)\Phi_i \\
= &\sum_{i=1}^{M}\eta_i\Phi_i - \sum_{i=1}^{M}F_i\Phi_i + \alpha\sum_{i=1}^{M}(X_{0,i} - \lambda^n_i)\Phi_i
\end{aligned}$$

where $P_h y^\delta = \sum_{i=1}^{M}\eta_i\Phi_i$, $P_h F(x^{h,\delta}_{n,\alpha}) = \sum_{i=1}^{M} F_i\Phi_i$ and $P_h(x_0 - x^{h,\delta}_{n,\alpha}) = \sum_{i=1}^{M}(X_{0,i} - \lambda^n_i)\Phi_i, i = 1, 2, \cdots M$. Then $x^{h,\delta}_{n+1,\alpha}$ is a solution of (3.2.1) if and only if $[\lambda^{n+1} - \lambda^n] = [\lambda^{n+1}_1 - \lambda^n_1, \lambda^{n+1}_2 - \lambda^n_2, ..., \lambda^{n+1}_M - \lambda^n_M]^T$ is the unique solution of

$$[M_M + \alpha B_M][\lambda^{n+1} - \lambda^n] = B_M[\eta - F^M + \alpha(X_0 - \lambda^M)]$$

where $M_M = (\langle F'(x^{h,\delta}_{n,\alpha})\Phi_i, \Phi_j\rangle), B_M = (\langle \Phi_i, \Phi_j\rangle) \;\; i, j = 1, 2, ..., M,$

$$F^M = [F_1, F_2, \cdots, F_M]^T,\ \eta = [\eta_1, \eta_2, \cdots, \eta_M]^T,$$

$$X_0 = [X_{0,1}, X_{0,2}, \cdots, X_{0,M}]^T \text{ and } \lambda^M = [\lambda^n_1, \lambda^n_2, \cdots, \lambda^n_M]^T.$$

3.6 NUMERICAL EXAMPLE

EXAMPLE 3.6.1. *Consider the following integral equation*

$$F(u) \equiv -\int\int_{\Omega} \frac{1}{[(x-x')^2+(y-y')^2+u^2(x',y')]^{1/2}} dx'dy' = f(x,y) \tag{3.6.1}$$

where $f(x,y) = \Delta g(x,y) + F(H)$ and $F : H'(\Omega) \subseteq L^2(\Omega) \to L^2(\Omega),\ \Omega = [0,m] \times [0,m]$.

The above equation satisfies the equation (3.2.15) (see (Vasin, 2013),(Vasin et al., 1996)).

The derivative of the operator F *at the point* $u_0(x,y)$ *is expressed by the formula*

$$F'(u_0)h = \int\int_{D} \frac{u_0(x',y')\,h(x',y')}{[(x-x')^2+(y-y')^2+(u_0(x',y'))^2]^{3/2}} dx'dy'. \tag{3.6.2}$$

Applying to the integral equations (3.6.1) two-dimensional analogy of rectangle's formula with uniform grid for every variable, we obtain the following system of nonlinear equations:

$$\sum_{i=1}^{m_2}\sum_{j=1}^{m_1} \frac{1}{[(x_k-x'_j)^2+(y_l-y'_i)^2+u^2(x'_j,y'_i)]^{1/2}} \Delta x \Delta y = f(x_k,y_l);$$

$k,l = 1,2,...,m$ *for the unknown vector* $\{u_{j,i} = u(x_j,y_i),\ i,j = 1,2,...,m\}$ *in vector-matrix form this system takes the form:*

$$F_M(u_M) = f_M, \tag{3.6.3}$$

where u_M, f_M *are vectors of dimension* $M = m^2$.

The discrete variant of the derivative $F'(u_0)$ *has the form*

$$\{F_n^0 h_n\}_{k,l} = \sum_{i=1}^{m}\sum_{j=1}^{m} \frac{\Delta x \Delta y\, u_0(x'_j,y'_i)h(x'_j,y'_i)}{[(x_k-x'_j)^2+(y_l-y'_i)^2+u_0^2(x'_j,y'_i)]^{3/2}}, \tag{3.6.4}$$

where $u_0(x,y) = H$ *is constant,* F_M^0 *is symmetric matrix, for which the* *component with member (k, l) is evaluated by formula (3.6.4).*

For our computation we have taken

$$\hat{u}(x, y) = 5 - 2exp^{-[(x/10-3.5)^2(y/10-2.5)^2]} - 3exp^{-[(x/10-5.5)^2(y/10-4.5)^2]},$$

where $\hat{u}(x, y)$ is given on the domain $D = \{0 \leq x \leq m, 0 \leq y \leq m\}$. Let $\Delta x = \Delta y = 1$, $M = m^2$, $\Delta = 0.25$, $H = 5$. We have taken $y^\delta = F(\hat{u}(x, y)) + \delta$ in our computations.

We choose orthonormal system of box function $\Phi_i(t, \tau) = \Psi_k(t)\Psi_l(\tau), i = (k-1)m + l,\ k, l = 1, 2, 3, .., m,\ i = 1, 2, ..., M(= m^2)$ where $\Psi_k(t), \Psi_l(\tau)$ are L_2-orthogonalized characteristic functions of the intervals $[\frac{k-1}{m}, \frac{k}{m}] \times [\frac{l-1}{m}, \frac{l}{m}]$ as a basis of X_M in $[0, 1] \times [0, 1]$.

In Table 3.1 the results of numerical experiments for different values of δ are presented. Here $x^{h,\delta}_{n,\alpha_k}$ is the numerical solution obtained by method (3.2.1); the relative error of solution and residual are

$$\Delta_1 = \frac{\|\hat{x} - x^{h,\delta}_{n,\alpha_k}\|}{\|x^{h,\delta}_{n,\alpha_k}\|}, \qquad \Delta_2 = \frac{\|F_n(x^{h,\delta}_{n,\alpha_k}) - y_n\|}{\|y_n\|}.$$

Table 3.1: Relative error and residual

$\delta + \epsilon_h$	α_k	m	Δ_1	Δ_2
0.1	0.3240		0.2281	1.9605
0.02	0.0648	35	0.2280	1.9606
0.01	0.0324		0.2277	1.9606
0.1	0.3240		0.2486	1.9598
0.02	0.0648	40	0.2484	1.9598
0.01	0.0324		0.2481	1.9599

The plots of the exact solution and approximate solution for different values of m and δ obtained is given in figure 3.1 to 3.8.

Figure 3.1: Exact solution for $m = 35$

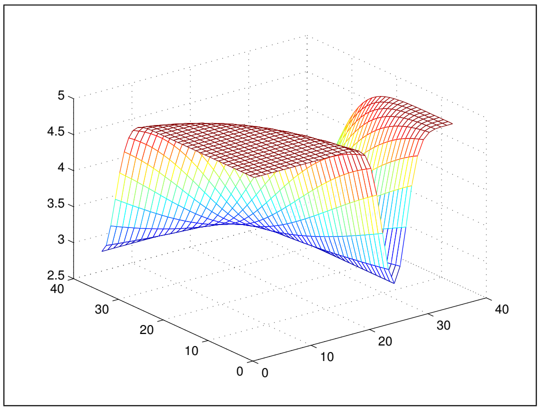


Figure 3.2: Approximate solution for $m = 35$ and $\delta = 0.1$

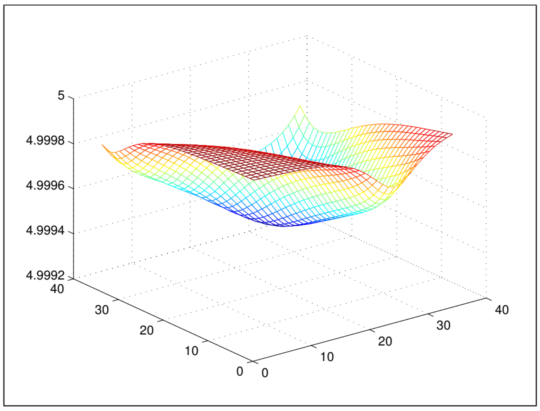


Figure 3.3: Approximate solution for $m = 35$ and $\delta = 0.02$

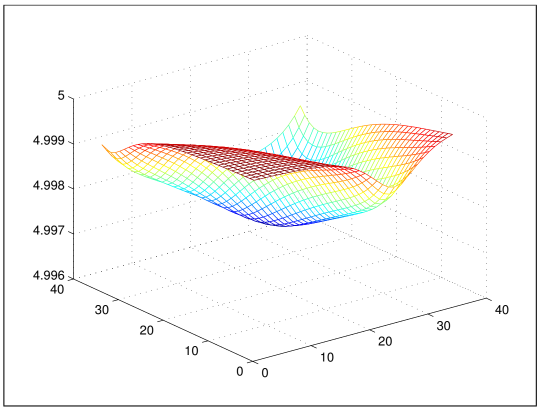


Figure 3.4: Approximate solution for $m = 35$ and $\delta = 0.01$

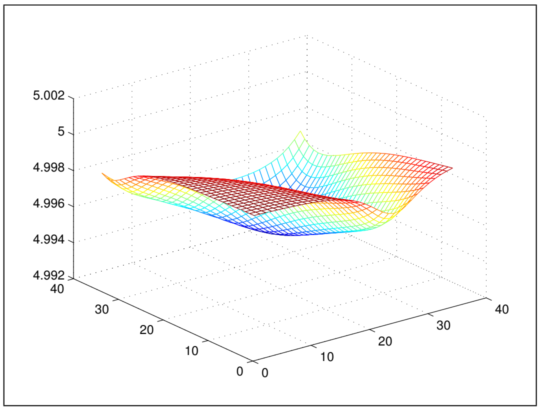


Figure 3.5: Exact solution for $m = 40$

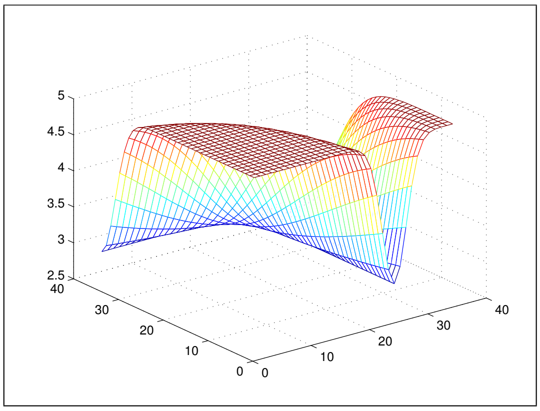


Figure 3.6: Approximate solution for $m = 40$ and $\delta = 0.1$

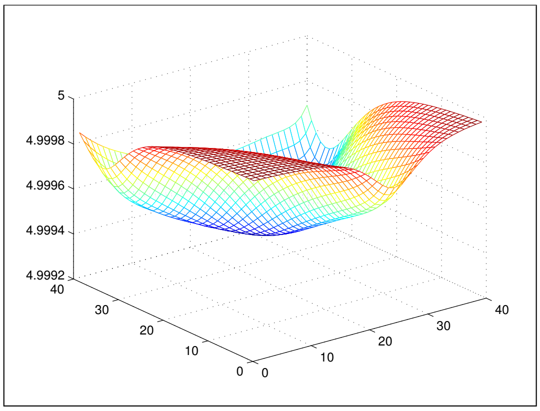


Figure 3.7: Approximate solution for $m = 40$ and $\delta = 0.02$

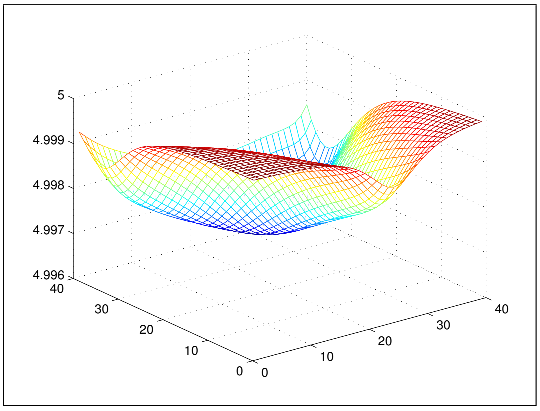
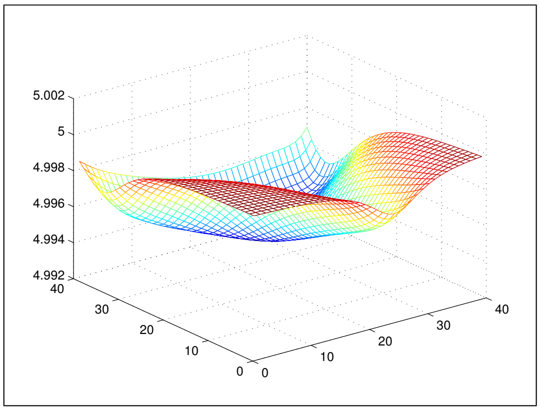


Figure 3.8: Approximate solution for $m = 40$ and $\delta = 0.01$



3.7 CONCLUSION

In this chapter we considered finite dimensional realization of the method considered in Chapter 2. We used adaptive scheme suggested by Pereverzev and Schock (2005) for choosing the regularization parameter. The numerical example justifies our theoretical results.

Chapter 4

LOCAL CONVERGENCE OF A TIKHONOV GRADIENT TYPE-METHOD UNDER WEAK CONDITIONS

In this chapter we consider the local convergence of a Tikhonov gradient type-method for approximating a solution $\hat{x}$ of the nonlinear ill-posed operator equation $F(x) = y$. In our convergence analysis, we use hypotheses only on the first Frèchet derivative of F in contrast to the higher derivatives used in the earlier studies.

4.1 INTRODUCTION

Throughout this chapter and next chapter we assume that $F : D(F) \subseteq X \to Y$ is weakly (sequentially) closed, continuous and Fréchet differentiable with convex domain $D(F)$. In Tikhonov regularization, one computes the global minimizer of the Tikhonov functional;

$$J_\alpha(x) = \min_{x \in D(F)} \|F(x) - y^\delta\|^2 + \alpha\|x - x_0\|^2 \tag{4.1.1}$$

where $\alpha > 0$ is a small regularization parameter. It is known (Scherzer et al. (1993)) that $J_\alpha(x)$ has a unique solution x_α^δ if F is weakly (sequentially) closed, continuous and Fréchet differentiable with convex domain $D(F)$.

Ramlau (2003) considered iterative method defined for $n = 0, 1, 2, \cdots$ by

$$x_{n+1}^{\delta} = x_n + \beta_n(F'(x_n^{\delta})^*(y^{\delta} - F(x_n^{\delta})) + \alpha_n(x_n^{\delta} - x_0)) \tag{4.1.2}$$

where β_n is a scaling parameter and α_n is the regularization parameter to obtain approximation for $\hat{x}$. The convergence analysis in Ramlau (2003) was carried out using the following assumptions $(\mathcal{A})$:

$(\mathcal{A}_1)$ F is twice Frèchet differentiable with continuous second derivative.

$(\mathcal{A}_2)$ The first derivative is Lipschitz continuous:

$$\|F'(x_1) - F'(x_2)\| \leq L\|x_1 - x_2\|.$$

$(\mathcal{A}_3)$ There exists $w \in Y$ with

$$\hat{x} - x_0 = F'(\hat{x})^* w.$$

$(\mathcal{A}_4)$ $\|w\| \leq q$ and $Lq \leq 0.241$.

And also assumed that Lipschitz constant L is explicitly given. Note that β_n in (4.1.2) is depending on the Lipschitz constant L and satisfying (see (4.15) in Ramlau (2003))

$$\beta_n \leq \min\{\frac{\gamma\alpha}{\| \bigtriangledown J_\alpha(x_n)\|^2}, \frac{4\gamma\alpha}{4K^2 + 4\alpha + 12q\alpha L + 4KL + L^2} \frac{(J_\alpha(x_n) - \phi_{min,n})}{\| \bigtriangledown J_\alpha(x_n)\|^2}\}.$$

Here $\phi_{min,n} = \min\{J_\alpha(x_n) + t\bigtriangledown J_\alpha(x_n) : t \in \mathbb{R}^+\}$, $K = \max\{L\|y^\delta - F(x_0)\| + \|F'(x_0)\|, \frac{2Lq}{\sqrt{1-Lq}} + (\frac{1}{1-Lq} + 1)\|F'(x_0)\|\}$ (see (3.14) in Ramlau (2003)) and γ is such that $\gamma + 3L\|w\| \leq \gamma + 3Lq < 1$ (see (3.20) in Ramlau (2003)).

The purpose of this chapter is to consider the local convergence of the modified form of the iterative procedure (4.1.2), but with a fixed β (a generic constant) and α instead of β_n and α_n. Here β is depending only on α and $\|F'(.)\|$. The modified Tikhonov Gradient-Type method(TGTM) is defined for $n = 1, 2, 3...$ by

$$u^\delta_{n+1,\alpha} = u^\delta_{n,\alpha} - \beta(F'(u^\delta_{n,\alpha})^*(F(u^\delta_{n,\alpha}) - y^\delta) + \alpha(u^\delta_{n,\alpha} - u_0)) \tag{4.1.3}$$

where u_0 is the initial approximation. The iterative procedure in (4.1.2) is a bit cumbersome than (4.1.3). Our approach in this chapter is two fold: (i) using hypothesis $(\mathcal{A}_2)$ we prove the convergence of $\{u^\delta_{n,\alpha}\}$ in (4.1.3) to u^δ_α, the unique solution of $J_\alpha(x)$. (ii) instead of $(\mathcal{A}_2)$, using two additional assumptions (Assumption 2.1.5 and 4.2.5) we prove the convergence of $\{u^\delta_{n,\alpha}\}$ in (4.1.3) to u^δ_α. We also obtain an error estimate for $\|u^\delta_{n,\alpha} - \hat{x}\|$ using a general source condition on $\hat{x} - u_0$ involving the operator $F'(\hat{x})$ (See Assumptions 4.3.1). Furthermore, our analysis is simpler than the analysis in Ramlau (2003). One of the main differences of our approach to that of Ramlau (2003) is that we fix the scaling and regularization parameters during the iteration.

The rest of the chapter is organized as follows: Section 4.2 deals with the convergence analysis of TGTM. In Section 4.3 we provide error bounds under certain general source conditions by choosing the regularization parameter by an a priori manner as well as by using an adaptive choice of the parameter proposed by Pereverzev and Schock (2005). Finally the chapter ends with a conclusion in Section 4.4.

4.2 CONVERGENCE ANALYSIS OF TGTM

We present local convergence analysis of method (4.1.3) in this section. Let $\delta_0 > 0$, $a_0 > 0$, $r_0 > 0$ and $r > 0$ be some constants with $L\delta_0 < a_0$, $\|u_0 - \hat{x}\| \leq r_0$ with $r_0 < \frac{\sqrt{\alpha}}{L} - \frac{\delta}{\sqrt{\alpha}}$ and

$$2(r_0 + 1) \leq r. \tag{4.2.1}$$

Let $M > 0$ be such that

$$\|F'(x)\| \leq M, \quad \forall x \in \overline{B(u_0, r)}, \tag{4.2.2}$$

$\delta \in (0, \delta_0]$ and $\alpha \in [\max\{L\delta, \delta^2\}, a_0]$. Further, let β, $q_{\alpha,\beta}$ be parameters such that

$$\beta = \frac{1}{M^2 + a_0} \tag{4.2.3}$$

and

$$q_{\alpha,\beta} = 1 - \alpha\beta + \beta L(\delta + \sqrt{\alpha} r_0) + \frac{\beta M L}{2} r. \tag{4.2.4}$$

Hereafter for simplicity we use the notation $u_n := u_{n,\alpha}^{\delta}$. We will be using the following well known results.

LEMMA 4.2.1 (Tautenhahn and Jin (2003)(Proposition 2.1)). *Let u_α^δ be the minimizer of* (4.1.1). *Then,*

$$\|u_\alpha^\delta - u_0\| \leq \frac{\delta}{\sqrt{\alpha}} + \|u_0 - \hat{x}\|.$$

LEMMA 4.2.2 (Kaltenbacher (1997)(Lemma 2.3)). *Let a_n be the sequence satisfying $0 \leq a_n \leq a$ and $\lim_{n\to 0} a_n \leq a$. Moreover, we assume that γ_n be the sequence satisfying*

$$0 \leq \gamma_{n+1} \leq a_n + b\gamma_n + c\gamma_n^2 \tag{4.2.5}$$

with $n \in \mathbb{N}$ and $\gamma_0 \geq 0$ that holds for some $b, c \geq 0$. Let γ' and $\bar{\gamma}$ be defined as $\gamma' = \frac{2a}{1-b+\sqrt{(1-b)^2-4ac}}$ and $\bar{\gamma} = \frac{1-b+\sqrt{(1-b)^2-4ac}}{2c}$. If $b + 2\sqrt{ac} < 1$ and if $\gamma_0 \leq \bar{\gamma}$, then

$$\gamma_n \leq \max\{\gamma_0, \gamma'\}.$$

THEOREM 4.2.3. *Let u_n be as in (4.1.3) and let $r < \frac{2(\alpha - L(\delta+\sqrt{\alpha} r_0))}{ML}$. Then for each $\delta \in (0, \delta_0]$, $\alpha \in [\max\{L\delta, \delta^2\}, a_0]$, the sequence $\{u_n\}$ is in $B(u_0, r)$ and converges to u_α^δ as $n \to \infty$. Further*

$$\|u_{n+1} - u_\alpha^\delta\| \leq q_{\alpha,\beta}^{n+1} \|u_0 - u_\alpha^\delta\|, \tag{4.2.6}$$

where $q_{\alpha,\beta}$ is as in (4.2.4).

Proof. Clearly, $u_0 \in \overline{B(u_0, r)}$. Let $A_n := \int_0^1 F'(u_\alpha^\delta + t(u_n - u_\alpha^\delta))dt$. By Lemma 4.2.1, we have $u_\alpha^\delta \in B(u_0, r)$, hence A_0 is well defined and $\|A_0\| \leq M$. Assume that for some $n > 0, u_n \in B(u_0, r)$ and A_n is well defined. Then, since u_α^δ satisfies the Euler equation

$$F'(u_\alpha^\delta)^*(F(u_\alpha^\delta) - y^\delta) + \alpha(u_\alpha^\delta - u_0) = 0 \tag{4.2.7}$$

we have,

$$\begin{aligned}
u_{n+1} - u_\alpha^\delta &= u_n - u_\alpha^\delta - \beta[F'(u_n)^*(F(u_n) - F(u_\alpha^\delta)) + \alpha(u_n - u_\alpha^\delta)] \\
&\quad +\beta[F'(u_\alpha^\delta)^* - F'(u_n)^*](F(u_\alpha^\delta) - y^\delta) \\
&= u_n - u_\alpha^\delta - \beta[F'(u_n)^* A_n + \alpha I](u_n - u_\alpha^\delta) \\
&\quad +\beta[F'(u_\alpha^\delta)^* - F'(u_n)^*](F(u_\alpha^\delta) - y^\delta) \\
&= u_n - u_\alpha^\delta - \beta[F'(u_n)^*(A_n - F'(u_n))](u_n - u_\alpha^\delta) \\
&\quad -\beta[F'(u_n)^* F'(u_n) + \alpha I](u_n - u_\alpha^\delta) \\
&\quad +\beta[F'(u_\alpha^\delta)^* - F'(u_n)^*](F(u_\alpha^\delta) - y^\delta) \\
&= [I - \beta(F'(u_n)^* F'(u_n) + \alpha I)](u_n - u_\alpha^\delta) \\
&\quad -\beta[F'(u_n)^*(A_n - F'(u_n))](u_n - u_\alpha^\delta) \\
&\quad +\beta[F'(u_\alpha^\delta)^* - F'(u_n)^*](F(u_\alpha^\delta) - y^\delta).
\end{aligned} \tag{4.2.8}$$

Now since $I - \beta(F'(u_n)^* F'(u_n) + \alpha I)$ is a positive self-adjoint operator,

$$\begin{aligned}
\|I - \beta(F'(u_n)^* F'(u_n) + \alpha I)\| &= \sup_{\|x\|=1} |\langle (I - \beta(F'(u_n)^* F'(u_n) + \alpha I))x, x\rangle| \\
&= \sup_{\|x\|=1} |(1 - \beta\alpha)\langle x, x\rangle - \beta\langle F'(u_n)^* F'(u_n)x, x\rangle| \\
&\leq 1 - \alpha\beta.
\end{aligned} \tag{4.2.9}$$

The last step follows from relation

$$\begin{aligned}
\beta|\langle F'(u_n)^* F'(u_n)x, x\rangle| \leq \beta\|F'(u_n)\|^2 &\leq \beta M^2 \leq \frac{1}{M^2 + \alpha} M^2 \\
&= 1 - \frac{\alpha}{M^2 + \alpha} \leq 1 - \beta\alpha.
\end{aligned}$$

Using $(\mathcal{A}_2)$, we have

$$\begin{aligned}
& \|\beta F'(u_n)^*(A_n - F'(u_n))(u_n - u_\alpha^\delta)\| \\
\leq\ & \|\beta F'(u_n)^* \left(\int_0^1 F'(u_\alpha^\delta + t(u_n - u_\alpha^\delta)) - F'(u_n) \right) dt (u_n - u_\alpha^\delta)\| \\
\leq\ & \beta \frac{ML}{2} \|u_n - u_\alpha^\delta\|^2
\end{aligned}$$

and

$$\begin{aligned}
& \|\beta[F'(u_\alpha^\delta)^* - F'(u_n)^*](F(u_\alpha^\delta) - y^\delta)\| \\
\leq\ & \beta\|F'(u_\alpha^\delta)^* - F'(u_n)^*\|\|F(u_\alpha^\delta) - y^\delta\| \\
=\ & \beta\|F'(u_\alpha^\delta) - F'(u_n)\|\|F(u_\alpha^\delta) - y^\delta\| \\
\leq\ & \beta L\|u_n - u_\alpha^\delta\|\|F(u_\alpha^\delta) - y^\delta\|.
\end{aligned}$$

Now using (4.1.1), we have

$$\|F(u_\alpha^\delta) - y^\delta\| \leq \delta + \sqrt{\alpha} r_0. \tag{4.2.10}$$

Hence,

$$\|u_{n+1} - u_\alpha^\delta\| \ \leq\ (1 - \alpha\beta + \beta L(\delta + \sqrt{\alpha} r_0))\|u_n - u_\alpha^\delta\| + \frac{\beta ML}{2}\|u_n - u_\alpha^\delta\|^2.$$

The above expression is of the form (4.2.5), where $a_n = 0, b = 1 - \alpha\beta + \beta L(\delta + \sqrt{\alpha} r_0)$, $\gamma_n = \|u_n - u_\alpha^\delta\|$ and $c = \frac{\beta ML}{2}$. We have by the condition on r_0, $b + 2\sqrt{ac} = b < 1$ and

$$\gamma_0 = \|u_0 - u_\alpha^\delta\| \leq \frac{1-b}{c} = \bar{\gamma}.$$

Hence by Lemma 4.2.2, we have

$$\begin{aligned}
\|u_{n+1} - u_\alpha^\delta\| \ &\leq\ (1 - \alpha\beta + \beta L(\delta + \sqrt{\alpha} r_0))\|u_n - u_\alpha^\delta\| + \frac{\beta ML}{2}\|u_0 - u_\alpha^\delta\|\|u_n - u_\alpha^\delta\| \\
&\leq\ (1 - \alpha\beta + \beta L(\delta + \sqrt{\alpha} r_0))\|u_n - u_\alpha^\delta\| + \frac{\beta ML}{2} r\|u_n - u_\alpha^\delta\| \\
&\leq\ q_{\alpha,\beta}\|u_n - u_\alpha^\delta\|. \qquad (4.2.11)
\end{aligned}$$

Thus, since $r < \frac{2(\alpha - L(\delta+\sqrt{\alpha}r_0))}{ML}$, we have $q_{\alpha,\beta} < 1$ and

$$\|u_{n+1} - u_\alpha^\delta\| < \|u_0 - u_\alpha^\delta\| \leq r$$

and

$$\|u_{n+1} - u_0\| < 2\|u_0 - u_\alpha^\delta\| \leq 2(r_0 + 1) \leq r$$

i.e., $u_{n+1} \in B(u_0, r)$. Also, for $0 \leq t \leq 1$,

$$\|u_\alpha^\delta + t(u_{n+1} - u_\alpha^\delta) - u_0\| = \|u_\alpha^\delta - u_0 + t(u_{n+1} - u_\alpha^\delta)\| < 2(r_0 + 1) \leq r.$$

Hence, $u_\alpha^\delta + t(u_{n+1} - u_\alpha^\delta) \in B(u_0, r)$ and A_{n+1} is well defined with $\|A_{n+1}\| \leq M$. Thus, by induction u_n is well defined and remains in $B(u_0, r)$ for each $n = 0, 1, 2, \cdots$. By letting $n \to \infty$ in (4.1.3), we obtain the convergence of u_n to u_α^δ. The estimate (4.2.6) now follows from (4.2.11). □

REMARK 4.2.4. *Note that the condition $r_0 < \frac{\sqrt{\alpha}}{L} - \frac{\delta}{\sqrt{\alpha}}$ is too restrictive. We can avoid this restriction by imposing some additional assumptions (see Assumptions 2.1.5 and 4.2.5). We also prove the convergence of (4.1.3) using the assumptions below.*

ASSUMPTION 4.2.5 (Scherzer et al. (1993))**.** *There exists K_1 such that for every $x, y \in B(u_0, r) \subseteq D(F)$ and $h \in X$, there exists element $\phi_1(x, y, h) \in X$ such that $[F'(x)^* - F'(y)^*]h = \phi_1(x, y, F'(y)^*h)$ with $\|\phi_1(x, y, F'(y)^*h)\| \leq K_1\|x - y\|\|F'(y)^*h\|$.*

Next, we shall give an example satisfying Assumptions 2.1.5 and 4.2.5.

EXAMPLE 4.2.6 (Engl et al. (1989), Scherzer et al. (1993))**.** *Consider the nonlinear Hammerstein operator*

$$(Fx)(t) = \int_0^1 k(t, \tau)g(\tau, x(\tau))d\tau,$$

with k continuous and g sufficiently smooth so that $F : H^1((0,1)) \to L^2((0,1))$ is Freˇchet differentiable with respect to x and

$$F'(x)h(t) = \int_0^1 k(t, \tau)g_x(\tau, x(\tau))h(\tau)d\tau.$$

Then F satisfies Assumptions 2.1.5 and 4.2.5 (see, Scherzer et al. (1993), Lemma 2.8).

Let $\delta_1 > 0$, $b_0 > 0$ and $\bar{r} > 0$ be some constants with $\delta_1^2 < b_0$ and

$$2(r_0 + 1) \leq \bar{r}. \tag{4.2.12}$$

Let $\delta \in (0, \delta_1]$ and $\alpha \in [\delta^2, b_0]$. Further, let β, $\bar{q}_{\alpha,\beta}$ be parameters such that

$$\beta = \frac{1}{M^2 + b_0} \tag{4.2.13}$$

and

$$\bar{q}_{\alpha,\beta} = 1 - \alpha\beta + \alpha\beta K_2 \bar{r} + \frac{\beta M^2 K_1}{2}\bar{r}. \tag{4.2.14}$$

THEOREM 4.2.7. *Let u_n be as in (4.1.3) and let $\bar{r} < \frac{2\alpha}{2\alpha K_1 + M^2 K_0}$. Then for each $\delta \in (0, \delta_1]$, $\alpha \in [\delta^2, b_0]$, the sequence $\{u_n\}$ is in $B(u_0, \bar{r})$ and converges to u_α^δ as $n \to \infty$. Further*

$$\|u_{n+1} - u_\alpha^\delta\| \leq \bar{q}_{\alpha,\beta}^{n+1} \|u_0 - u_\alpha^\delta\| \tag{4.2.15}$$

where $\bar{q}_{\alpha,\beta}$ is as in (4.2.14).

Proof. Clearly, $u_0 \in \overline{B(u_0, \bar{r})}$. Let $A_n := \int_0^1 F'(u_\alpha^\delta + t(u_n - u_\alpha^\delta))dt$. By Lemma 4.2.1, we have $u_\alpha^\delta \in B(u_0, \bar{r})$, hence A_0 is well defined and $\|A_0\| \leq M$. Assume that for some $n > 0, u_n \in B(u_0, \bar{r})$ and A_n is well defined. Using (4.2.8), Assumptions 2.1.5 and 4.2.5 we have

$$\begin{aligned}
u_{n+1} - u_\alpha^\delta &= [I - \beta(F'(u_n)^* F'(u_n) + \alpha I)](u_n - u_\alpha^\delta) \\
&\quad -\beta[F'(u_n)^* \int_0^1 F'(u_n)\phi(u_\alpha^\delta + t(u_n - u_\alpha^\delta), u_n, u_n - u_\alpha^\delta)]dt \\
&\quad -\beta\phi_1(u_n, u_\alpha^\delta, F'(u_\alpha^\delta)^*(F(u_\alpha^\delta) - y^\delta)) \\
&= [I - \beta(F'(u_n)^* F'(u_n) + \alpha I)](u_n - u_\alpha^\delta) \\
&\quad -\beta[F'(u_n)^* F'(u_n) \int_0^1 \phi(u_\alpha^\delta + t(u_n - u_\alpha^\delta), u_n, u_n - u_\alpha^\delta)dt \\
&\quad -\beta\phi_1(u_n, u_\alpha^\delta, -\alpha(u_\alpha^\delta - u_0)).
\end{aligned}$$

Hence, using (4.2.9) we have

$$\begin{aligned}\|u_{n+1}-u_\alpha^\delta\| &\le (1-\alpha\beta)\|u_n-u_\alpha^\delta\| + \beta M^2K_0\|u_n-u_\alpha^\delta\|^2\int_0^1(1-t)dt\\ &\quad +\beta K_1\alpha\|u_n-u_\alpha^\delta\|\|u_\alpha^\delta-u_0\|\\ &\le (1-\alpha\beta+\alpha\beta K_1\|u_\alpha^\delta-u_0\|)\|u_n-u_\alpha^\delta\| + \frac{\beta M^2K_0}{2}\|u_n-u_\alpha^\delta\|^2\\ &\le (1-\alpha\beta+\alpha\beta K_1\bar{r})\|u_n-u_\alpha^\delta\| + \frac{\beta M^2K_0}{2}\|u_n-u_\alpha^\delta\|^2.\end{aligned}$$

The above expression is of the form (4.2.5), where $a_n=0, b=1-\alpha\beta+\alpha\beta K_1\bar{r}$, $\gamma_n=\|u_n-u_\alpha^\delta\|$ and $c=\frac{\beta M^2K_0}{2}$. We have by the condition on $\bar{r}$, $b+2\sqrt{ac}=b<1$ and

$$\gamma_0=\|u_0-u_\alpha^\delta\|\le\frac{1-b}{c}=\bar{\gamma}.$$

Hence by Lemma 4.2.2, we have

$$\begin{aligned}\|u_{n+1}-u_\alpha^\delta\| &\le (1-\alpha\beta+\alpha\beta K_1\|u_0-u_\alpha^\delta\|+\frac{\beta M^2K_0}{2}\|u_0-u_\alpha^\delta\|)\|u_n-u_\alpha^\delta\|\\ &\le (1-\alpha\beta+\alpha\beta K_1\bar{r}+\frac{\beta M^2K_0}{2}\bar{r})\|u_n-u_\alpha^\delta\|\\ &\le \bar{q}_{\alpha,\beta}\|u_n-u_\alpha^\delta\|. \qquad (4.2.16)\end{aligned}$$

Thus, since $\bar{q}_{\alpha,\beta}<1$, we have

$$\|u_{n+1}-u_\alpha^\delta\|<\|u_0-u_\alpha^\delta\|\le\bar{r}$$

and

$$\|u_{n+1}-u_0\|<2\|u_0-u_\alpha^\delta\|\le 2(r_0+1)\le\bar{r}$$

i.e., $u_{n+1}\in B(u_0,\bar{r})$. Also, for $0\le t\le 1$,

$$\|u_\alpha^\delta+t(u_{n+1}-u_\alpha^\delta)-u_0\|=\|u_\alpha^\delta-u_0+t(u_{n+1}-u_\alpha^\delta)\|<2(r_0+1)\le\bar{r}.$$

Hence, $u_\alpha^\delta+t(u_{n+1}-u_\alpha^\delta)\in B(u_0,\bar{r})$ and A_{n+1} is well defined with $\|A_{n+1}\|\le M$. Thus, by induction u_n is well defined and remains in $B(u_0,\bar{r})$ for each

$n = 0, 1, 2, \cdots$. By letting $n \to \infty$ in (4.1.3), we obtain the convergence of u_n to u_α^δ. The estimate (4.2.15) now follows from (4.2.16). □

4.3 ERROR BOUNDS UNDER SOURCE CONDITIONS

Hereafter, we use the estimate in Theorem 4.2.7 to obtain error estimate for $\|u_{n,\alpha}^\delta - \hat{x}\|$. Similar result can be obtained using the estimate in Theorem 4.2.3. For the convenience of the convergence analysis that follows, we use the following well known assumption from Semenova (2010).

ASSUMPTION 4.3.1. *There exists a continuous, strictly monotonically increasing function* $\Phi : (0, \bar{a}] \to (0, \infty)$ *with* $\bar{a} \geq \|F'(\hat{x})\|^2$ *satisfying*

(i) $\lim_{\lambda \to 0} \Phi(\lambda) = 0$.

(ii) $\sup_{\lambda \geq 0} \frac{\alpha \Phi(\lambda)}{\lambda + \alpha} \leq \Phi(\alpha), \forall \alpha \in (0, \bar{a}]$.

(iii) There exists $v \in X$ *with* $\|v\| \leq 1$ *such that*

$$u_0 - \hat{x} = \Phi(F'(\hat{x})^* F'(\hat{x}))v.$$

THEOREM 4.3.2. *Let* u_α^δ *be the minimizer of (4.1.1) and let*

$$\bar{r} < \min\{\frac{2\alpha}{2\alpha K_1 + M^2 K_0}, \frac{1}{2K_0 + K_1}\}.$$

Then

$$\|u_\alpha^\delta - \hat{x}\| \leq \frac{1}{1 + K_0 + K_1 - \frac{\bar{r}}{2}(2K_0 + K_1)}[\frac{\delta}{\sqrt{\alpha}} + \Phi(\alpha)].$$

Proof. Let $\hat{M} = \int_0^1 F'(\hat{x} + t(u_\alpha^\delta - \hat{x}))dt$ and $A = F'(u_\alpha^\delta)$. Then from (4.2.7) we have

$$(A^* \hat{M} + \alpha I)(u_\alpha^\delta - \hat{x}) = A^*(y^\delta - y) + \alpha(u_0 - \hat{x})$$

and

$$\begin{aligned} u_\alpha^\delta - \hat{x} &= (A^*A + \alpha I)^{-1} A^*(A - \hat{M})(u_\alpha^\delta - \hat{x}) + (A^*A + \alpha I)^{-1} A^*(y^\delta - y) \\ &\quad + (A^*A + \alpha I)^{-1} \alpha(u_0 - \hat{x}). \end{aligned}$$

Therefore

$$\|u_\alpha^\delta - \hat{x}\| \le \|\Gamma_1\| + \frac{\delta}{\sqrt{\alpha}} + \|\Gamma_2\| \qquad (4.3.1)$$

where $\Gamma_1 = (A^*A+\alpha I)^{-1}A^*(A-\hat{M})(u_\alpha^\delta - \hat{x})$ and $\Gamma_2 = (A^*A+\alpha I)^{-1}\alpha(u_0-\hat{x})$. Using definition of $\hat{M}$ and Assumption 2.1.5, we have in turn

$$\begin{aligned}\Gamma_1 &= (A^*A+\alpha I)^{-1}A^*[F'(u_\alpha^\delta) - \int_0^1 F'(\hat{x}+t(u_\alpha^\delta-\hat{x}))dt](u_\alpha^\delta-\hat{x})\\ &= (A^*A+\alpha I)^{-1}A^*[\int_0^1 F'(u_\alpha^\delta)\\ &\quad -F'(\hat{x}+t(u_\alpha^\delta-\hat{x}))dt](u_\alpha^\delta-\hat{x}) \qquad (4.3.2)\\ &= -(A^*A+\alpha I)^{-1}A^*\int_0^1 A\phi(\hat{x}+t(u_\alpha^\delta-\hat{x}), u_\alpha^\delta, u_\alpha^\delta-\hat{x})dt.\end{aligned}$$

Now, by using triangle inequality, Lemma 4.2.1 and the definition of $\bar{r}$, we have

$$\begin{aligned}\|\Gamma_1\| &\le \frac{K_0}{2}\|u_\alpha^\delta-\hat{x}\|^2\\ &\le \frac{K_0\bar{r}}{2}\|u_\alpha^\delta-\hat{x}\|. \qquad (4.3.3)\end{aligned}$$

Let $\hat{A} := F'(\hat{x})$. Then using Assumptions 2.1.5, 4.2.5 and 4.3.1, we have in turn

$$\begin{aligned}\|\Gamma_2\| &= \|[(A^*A+\alpha I)^{-1} - (\hat{A}^*\hat{A}+\alpha I)^{-1}]\alpha(u_0-\hat{x})\\ &\quad +(\hat{A}^*\hat{A}+\alpha I)^{-1}\alpha(u_0-\hat{x})\|\\ &\le \|[(A^*A+\alpha I)^{-1}(\hat{A}^*\hat{A}-A^*A)(\hat{A}^*\hat{A}+\alpha I)^{-1}]\alpha(u_0-\hat{x})\|\\ &\quad +\|\hat{A}^*\hat{A}+\alpha I)^{-1}\alpha(u_0-\hat{x})\|\\ &\le \|[(A^*A+\alpha I)^{-1}((\hat{A}^*-A^*)\hat{A}-A^*(A-\hat{A}))(\hat{A}^*\hat{A}+\alpha I)^{-1}]\alpha(u_0-\hat{x})\|\\ &\quad +\|(\hat{A}^*\hat{A}+\alpha I)^{-1}\alpha(u_0-\hat{x})\|\end{aligned}$$

$$
\begin{aligned}
&\leq \|(A^*A+\alpha I)^{-1}((\hat{A}^*-A^*)\hat{A}(\hat{A}^*\hat{A}+\alpha I)^{-1}\alpha(u_0-\hat{x})\| \\
&\quad +\|(A^*A+\alpha I)^{-1}A^*(A-\hat{A}))(\hat{A}^*\hat{A}+\alpha I)^{-1}\alpha(u_0-\hat{x})\| \\
&\quad +\|(\hat{A}^*\hat{A}+\alpha I)^{-1}\alpha(u_0-\hat{x})\| \qquad (4.3.4) \\
&\leq \|(A^*A+\alpha I)^{-1}\|\|\phi_1(u_\alpha^\delta,\hat{x},\hat{A}^*\hat{A}(\hat{A}^*\hat{A}+\alpha I)^{-1}\alpha(u_0-\hat{x}))\| \\
&\quad +\|(A^*A+\alpha I)^{-1}A^*A\|\|\phi(u_\alpha^\delta,\hat{x},(\hat{A}^*\hat{A}+\alpha I)^{-1}\alpha(u_0-\hat{x}))\| \\
&\quad +\|(\hat{A}^*\hat{A}+\alpha I)^{-1}\alpha(u_0-\hat{x})\| \\
&\leq K_1\|u_\alpha^\delta-\hat{x}\|\|u_0-\hat{x}\|+K_0\|u_\alpha^\delta-\hat{x}\|\|u_0-\hat{x}\|+\Phi(\alpha)\|v\| \\
&\leq (K_0+K_1)r_0\|u_\alpha^\delta-\hat{x}\|+\Phi(\alpha)\|v\| \\
&\leq (K_0+K_1)(\frac{\bar{r}}{2}-1)\|u_\alpha^\delta-\hat{x}\|+\Phi(\alpha)\|v\|. \qquad (4.3.5)
\end{aligned}
$$

The result now follows from (4.3.1), (4.3.3) and (4.3.5). □

REMARK 4.3.3. *If we use ($\mathcal{A}_2$), instead of Assumptions 2.1.5 and 4.2.5, then by (4.3.2) we have $\|\Gamma_1\| \leq \frac{Lr}{2\sqrt{\alpha}}\|u_\alpha^\delta-\hat{x}\|$ and by (4.3.4) we have $\|\Gamma_2\| \leq \frac{2Lr_0}{\sqrt{\alpha}}\|u_\alpha^\delta-\hat{x}\|+\phi(\alpha)$. Hence in this case we have*

$$\|u_\alpha^\delta-\hat{x}\| \leq \frac{1}{1-\frac{4Lr_0+Lr}{2\sqrt{\alpha}}}\Phi(\alpha)$$

provided $4Lr_0+Lr<2\sqrt{\alpha}$.

Combining the estimates in Theorems 4.2.7 and 4.3.2 we have the following theorem.

THEOREM 4.3.4. *Let u_n be as in (4.1.3) and let the assumptions in Theorems 4.2.7 and 4.3.2 be satisfied. Then we have*

$$\|u_{n+1,\alpha}^\delta-\hat{x}\| \leq \bar{q}_{\alpha,\beta}^{n+1}\bar{r}+\frac{1}{1+K_0+K_1-\frac{\bar{r}}{2}(2K_0+K_1)}(\frac{\delta}{\sqrt{\alpha}}+\Phi(\alpha)).$$

Let

$$n_\delta=\min\{n:\bar{q}_{\alpha,\beta}^n \leq \frac{\delta}{\sqrt{\alpha}}\}. \qquad (4.3.6)$$

THEOREM 4.3.5. *Let u_n be as in (4.1.3) and let the assumptions in The-*

orem 4.3.4 be satisfied. Let n_δ be as in (4.3.6). Then

$$\|u^{\delta}_{n_\delta,\alpha} - \hat{x}\| \leq \bar{C}(\frac{\delta}{\sqrt{\alpha}} + \Phi(\alpha)) \tag{4.3.7}$$

where $\bar{C} = \bar{r} + \frac{1}{1+K_0+K_1-\frac{\bar{r}}{2}(2K_0+K_1)}$.

4.3.1 A priori choice of the parameter

Note that the estimate $\frac{\delta}{\sqrt{\alpha}} + \Phi(\alpha)$ in (4.3.7) is of optimal order for the choice $\alpha := \alpha_\delta$ which satisfies, $\frac{\delta}{\sqrt{\alpha}} = \Phi(\alpha)$. Now using the function $\psi(\lambda) := \lambda\sqrt{\Phi^{-1}(\lambda)}, 0 < \lambda \leq \|F'(\hat{x})\|^2$, we have $\delta = \sqrt{\alpha_\delta}\Phi(\alpha_\delta) = \psi(\Phi(\alpha_\delta))$ so that $\alpha_\delta = \Phi^{-1}(\psi^{-1}(\delta))$. Hence by Theorem 4.3.5 we have the following.

THEOREM 4.3.6. *Let $\psi(\lambda) := \lambda\sqrt{\Phi^{-1}(\lambda)}$ for $0 < \lambda \leq \|F'(\hat{x})\|^2$, and let the assumptions in Theorem 4.3.5 holds. For $\delta \in (0, \delta_0]$, let $\alpha := \alpha_\delta = \Phi^{-1}(\psi^{-1}(\delta))$ and let n_δ be as in (4.3.6). Then*

$$\|u^{\delta}_{n_\delta,\alpha} - \hat{x}\| = O(\psi^{-1}(\delta)).$$

4.3.2 Balancing Principle

Observe that the a priori choice of the parameter could be achieved only in the ideal situation when the function ψ is known. The point is that the best function ψ measuring the rate of convergence in Theorem 4.3.5 is usually unknown. Therefore in practical applications different parameters $\alpha = \alpha_i$ are often selected from the finite set

$$D_N := \{\alpha_i = \mu^i\alpha_0 < 1, i = 1, 2, ..., N\},$$

where $\alpha_0 = \delta^2$ (see Semenova (2010)) and $\mu > 1$ and corresponding elements $u^{\delta}_{n,\alpha_i}, i = 1, 2, ..., N$ are studied. Let

$$n_i := \min\{n : \bar{q}^n_{\alpha,\beta} \leq \frac{\delta}{\sqrt{\alpha_i}}\}$$

and let $u^\delta_{\alpha_i} := u^\delta_{n_i,\alpha_i}$. Then from Theorem 4.3.5, we have

$$\|u^\delta_{\alpha_i} - \hat{x}\| \leq \bar{C}(\frac{\delta}{\sqrt{\alpha_i}} + \Phi(\alpha_i)), \forall i = 1, 2, ..., N.$$

The main result of this section is the following theorem, proof of which is analogous to the proof of Theorem 4.4 in George and Nair (2008).

THEOREM 4.3.7. *Assume that there exists $i \in \{0, 1, ...N\}$ such that $\Phi(\alpha_i) \leq \frac{\delta}{\sqrt{\alpha_i}}$. Let assumptions of Theorem 4.3.4 be satisfied and let*

$$l := \max\left\{i : \Phi(\alpha_i) \leq \frac{\delta}{\sqrt{\alpha_i}}\right\} < N,$$

$$k := \max\left\{i : \forall j = 1, 2, ..., i-1; \|u^\delta_{\alpha_i} - u^\delta_{\alpha_j}\| \leq 4\bar{C}\frac{\delta}{\sqrt{\alpha_j}}\right\}$$

where $\bar{C}$ is as in Theorem 4.3.5. Then $l \leq k$ and

$$\|u^\delta_{\alpha_i} - \hat{x}\| \leq 6\bar{C}\mu\psi^{-1}(\delta).$$

4.4 CONCLUSION

In this chapter we considered a modified Tikhonov gradient type-method for approximately solving the nonlinear ill-posed operator equation $F(x) = y$ where $F : D(F) \subseteq X \to Y$ is a nonlinear operator between the Hilbert spaces X and Y. The method is a modified form of the Tikhonov Gradient method considered in Ramlau (2003). The assumptions used for the convergence analysis in Theorem 4.2.3 (we used only $(\mathcal{A}_2)$ with an additional assumption on the initial guess) is weaker than that of Ramlau (2003). We use the adaptive method considered by in Pereverzev and Schock (2005) for choosing the regularization parameter.

In the next chapter we consider the finite dimensional realization and the implementation of the method considered in this chapter.

Chapter 5

FINITE DIMENSIONAL REALIZATION OF A TIKHONOV GRADIENT TYPE-METHOD UNDER WEAK CONDITIONS

In Chapter 4 we considered a Tikhonov gradient type iterative method for obtaining approximate solution to operator equation of the form $F(x) = y$ where $F : D(F) \in X \to Y$ is a non linear operator between Hilbert space X and Y. In this chapter we consider projection techniques to obtain the finite dimensional realization of a Tikhonov gradient type-method considered in Chapter 4 for approximating a solution $\hat{x}$ of the nonlinear ill-posed operator equation $F(x) = y$. The regularization parameter is chosen according to the adaptive method considered by Pereverzev and Schock (2005). We also derive optimal stopping conditions on the number of iterations necessary for obtaining the optimal order of convergence. Using two numerical examples we compare our results with an existing method to justify the theoretical results.

5.1 INTRODUCTION

For practical problems, we need to find an approximate solution in a finite dimensional subspace of X. Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of finite dimensional subspace of X and let

$$J_{\alpha,i}(x) = \min_{x \in X_i} \{\|F(x) - y^\delta\|^2 + \alpha\|x - P_h u_0\|^2\} \tag{5.1.1}$$

where $\alpha > 0$, and $P_h(h = \frac{1}{i})$ is the projection onto X_i. Then $J_{\alpha,i}(x)$ has a unique solution $u_\alpha^{h,\delta}$ in X_i given by

$$P_h F'(u_\alpha^{h,\delta})^* P_h(F(u_\alpha^{h,\delta}) - y^\delta) + \alpha(u_\alpha^{h,\delta} - u_0^{h,\delta}) = 0 \tag{5.1.2}$$

where $u_0^{h,\delta} := P_h u_0$. Regularization of ill-posed problems by projection methods can be found in literature, for e.g in Groetsch and Neubauer (1988), Kaltenbacher et al. (2008), Kirsch (2011). Our aim in this chapter is to obtain an approximation for u_α^δ, in the finite dimensional space $R(P_h)$ of X. Here $\{P_h\}_{h>0}$ is a family of orthogonal projections of X onto $R(P_h)$, the range of P_h i.e., X_i. We need the following conditions to prove our results. Let

$$\epsilon_h := \|F'(.)(I - P_h)\|,$$

$$b_h := \|(I - P_h)\hat{x}\|,$$

and

$$c_h := \|(I - P_h)u_0\|.$$

We assume that $\lim_{h\to 0} \epsilon_h = 0$, $\lim_{h\to 0} b_h = 0$ and $\lim_{h\to 0} c_h = 0$. The above assumptions are satisfied if $P_h \to I$ point wise and if $F'(.)$ is compact operator. Further we assume that there exist $\epsilon_0 > 0, b_0 > 0, c_0 > 0$ and $\delta_0 > 0$ such that $\epsilon_h < \epsilon_0, b_h < b_0, c_h < c_0$ and $\delta < \delta_0$.

Finite dimensional version of Tikhonov Gradient Type method(FDTGTM) is defined for each $n = 1, 2, 3, \cdots$ by

$$u_{n+1,\alpha}^{h,\delta} = u_{n,\alpha}^{h,\delta} - \beta P_h[F'(u_{n,\alpha}^{h,\delta})^* P_h(F(u_{n,\alpha}^{h,\delta}) - y^\delta) + \alpha(u_{n,\alpha}^{h,\delta} - u_0^{h,\delta})] \tag{5.1.3}$$

where $u_{0,\alpha}^{h,\delta} = P_h u_0$ is an approximation for $u_\alpha^{h,\delta}$. Note that no inversion of the operator is involved in the method (5.1.3). This is the main advantage of our method over the existing iterative regularization methods, studied extensively in literature, Argyros and George (2013), Argyros et al. (2014), Bakushinskii (1992), Bauer et al. (2009), Blaschke et al. (1997), George and Nair (2008), Kaltenbacher et al. (2008), Vasin (2013), Vasin and George (2014).

The rest of the chapter is organized as follows: The convergence analysis of the method is given in Section 5.2. In Section 5.3 we provide error bounds under certain general source conditions by choosing the regularization parameter by an a priori manner as well as by using an adaptive choice of the parameter proposed by Pereverzev and Schock (2005). In Chapter 3, we considered the iterative method defined for $n = 1, 2, 3, \cdots$ by (3.2.1). In section 5.4 we compare the numerical results of method (5.1.3) and (3.2.1).

5.2 CONVERGENCE ANALYSIS OF FDTGTM

Let $r > 0$ be some constant such that

$$2\Big(\max\{1, \|\hat{x}\|\} + r_0\Big) + c_0 \leq r$$

where $\|\hat{x} - u_0\| \leq r_0$ and $(\delta_0 + \epsilon_0)^2 < \alpha$. Let $\delta \in (0, \delta_0]$, $\alpha \in [(\delta + \epsilon_h)^2, a_0)$, $(\delta_0 + \epsilon_0)^2 \leq a_0$ and $\|F'(x)\| \leq M, \ \ \forall x \in D(F)$. Further, let β, $q^h_{\alpha,\beta}$ be parameters such that

$$\beta = \frac{1}{M^2 + a_0}$$

and

$$q^h_{\alpha,\beta} = 1 - \alpha\beta + \alpha\beta K_1 r + \frac{\beta M^2 K_0}{2} r. \tag{5.2.1}$$

We need the following lemma to prove our results:

LEMMA 5.2.1. *Let $u_\alpha^{h,\delta}$ be the minimizer of* (5.1.1). *Then,*

$$\|u_\alpha^{h,\delta} - P_h u_0\| \leq \frac{\epsilon_h \|\hat{x}\| + \delta}{\sqrt{\alpha}} + \|u_0 - \hat{x}\|.$$

Proof. Since $u_\alpha^{h,\delta}$ is the minimizer of (5.1.1), we have

$$
\begin{aligned}
& \|F(u_\alpha^{h,\delta}) - y^\delta\|^2 + \alpha\|u_\alpha^{h,\delta} - P_h u_0\|^2 \\
\leq\ & \|F(P_h\hat{x}) - y^\delta\|^2 + \alpha\|P_h(\hat{x} - u_0)\|^2 \\
\leq\ & \|F(P_h\hat{x}) - F(\hat{x}) + F(\hat{x}) - y^\delta\|^2 + \alpha\|\hat{x} - u_0\|^2 \\
\leq\ & \left(\|\int_0^1 F'(\hat{x} + t(P_h\hat{x} - \hat{x}))dt(P_h\hat{x} - \hat{x})\| + \delta\right)^2 \\
& +\alpha\|\hat{x} - u_0\|^2 \\
\leq\ & \left(\|\int_0^1 F'(\hat{x} + t(P_h\hat{x} - \hat{x}))(P_h - I)\hat{x}dt\| + \delta\right)^2 + \alpha\|\hat{x} - u_0\|^2 \\
\leq\ & (\epsilon_h\|\hat{x}\| + \delta)^2 + \alpha\|\hat{x} - u_0\|^2.
\end{aligned}
$$

Hence,

$$
\|u_\alpha^{h,\delta} - P_h u_0\| \leq \frac{\epsilon_h\|\hat{x}\| + \delta}{\sqrt{\alpha}} + \|\hat{x} - u_0\|.
$$

□

THEOREM 5.2.2. *Let $u_{n,\alpha}^{h,\delta}$ be as in (5.1.3) and let $r < \frac{2\alpha}{2\alpha K_1 + M^2 K_0}$. Then for each $\delta \in (0, \delta_0]$, $\alpha \in ((\delta + \epsilon_h)^2, a_0], \epsilon_h \leq \epsilon_0$ the sequence $\{u_{n,\alpha}^{h,\delta}\}$ is in $B(u_0, r) \cap R(P_h)$ and converges to $u_\alpha^{h,\delta}$ as $n \to \infty$. Further*

$$
\|u_{n+1,\alpha}^{h,\delta} - u_\alpha^{h,\delta}\| \leq (q_{\alpha,\beta}^h)^{n+1}\|P_h u_0 - u_\alpha^{h,\delta}\| \tag{5.2.2}
$$

where $q_{\alpha,\beta}^h$ is as in (5.2.1).

Proof. Let $A_n^h := \int_0^1 F'(u_\alpha^{h,\delta} + t(u_{n,\alpha}^{h,\delta} - u_\alpha^{h,\delta}))dt$. Since $u_\alpha^{h,\delta}$ satisfies (5.1.2) we have,

$$
\begin{aligned}
u_{n+1,\alpha}^{h,\delta} - u_\alpha^{h,\delta} &= u_{n,\alpha}^{h,\delta} - u_\alpha^{h,\delta} - \beta[P_h F'(u_{n,\alpha}^{h,\delta})^* P_h(F(u_{n,\alpha}^{h,\delta}) - F(u_\alpha^{h,\delta})) \\
&\quad +\alpha P_h(u_{n,\alpha}^{h,\delta} - u_\alpha^{h,\delta})] \\
&\quad +\beta P_h[F'(u_\alpha^{h,\delta})^* - F'(u_{n,\alpha}^{h,\delta})^*]P_h(F(u_\alpha^{h,\delta}) - y^\delta) \\
&= u_{n,\alpha}^{h,\delta} - u_\alpha^{h,\delta} - \beta P_h[F'(u_{n,\alpha}^{h,\delta})^* A_n^h + \alpha P_h](u_{n,\alpha}^{h,\delta} - u_\alpha^{h,\delta}) \\
&\quad +\beta P_h[F'(u_\alpha^{h,\delta})^* - F'(u_{n,\alpha}^{h,\delta})^*]P_h(F(u_\alpha^{h,\delta}) - y^\delta)
\end{aligned}
$$

$$
\begin{aligned}
&= u_{n,\alpha}^{h,\delta} - u_{\alpha}^{h,\delta} - \beta P_h[F'(u_{n,\alpha}^{\delta})^*(A_n^h - F'(u_{n,\alpha}^{h,\delta}))](u_{n,\alpha}^{h,\delta} - u_{\alpha}^{h,\delta}) \\
&\quad -\beta P_h[F'(u_{n,\alpha}^{h,\delta})^* F'(u_{n,\alpha}^{h,\delta}) + \alpha P_h] P_h (u_{n,\alpha}^{h,\delta} - u_{\alpha}^{h,\delta}) \\
&\quad +\beta P_h[F'(u_{\alpha}^{h,\delta})^* - F'(u_{n,\alpha}^{h,\delta})^*](F(u_{\alpha}^{h,\delta}) - y^{\delta}) \\
&= [I - \beta(P_h F'(u_{n,\alpha}^{h,\delta})^* F'(u_{n,\alpha}^{h,\delta}) P_h + \alpha I)](u_{n,\alpha}^{h,\delta} - u_{\alpha}^{h,\delta}) \\
&\quad -\beta P_h[F'(u_{n,\alpha}^{h,\delta})^*(A_n^h - F'(u_{n,\alpha}^{h,\delta}))](u_{n,\alpha}^{h,\delta} - u_{\alpha}^{h,\delta}) \\
&\quad +\beta P_h[F'(u_{\alpha}^{h,\delta})^* - F'(u_{n,\alpha}^{h,\delta})^*] P_h (F(u_{\alpha}^{h,\delta}) - y^{\delta}). \qquad (5.2.3)
\end{aligned}
$$

Now since $I - \beta(P_h F'(u_{n,\alpha}^{h,\delta})^* F'(u_{n,\alpha}^{h,\delta}) P_h + \alpha I)$ is a positive self-adjoint operator,

$$
\begin{aligned}
&\|I - \beta(P_h F'(u_{n,\alpha}^{h,\delta})^* F'(u_{n,\alpha}^{h,\delta}) P_h + \alpha I)\| \\
&= \sup_{\|x\|=1} |\langle (I - \beta(P_h F'(u_{n,\alpha}^{h,\delta})^* F'(u_{n,\alpha}^{h,\delta}) P_h + \alpha I))x, x\rangle| \\
&= \sup_{\|x\|=1} |(1 - \beta\alpha)\langle x, x\rangle - \beta\langle P_h F'(u_{n,\alpha}^{h,\delta})^* F'(u_{n,\alpha}^{h,\delta}) P_h x, x\rangle| \\
&\leq 1 - \alpha\beta. \qquad (5.2.4)
\end{aligned}
$$

The last step follows from relation

$$
\begin{aligned}
\beta|\langle P_h F'(u_{n,\alpha}^{h,\delta})^* F'(u_{n,\alpha}^{h,\delta}) P_h x, x\rangle| &\leq \beta\|P_h F'(u_{n,\alpha}^{h,\delta})\|^2 \\
&\leq \beta M^2 \leq \frac{1}{M^2 + \alpha} M^2 \\
&= 1 - \frac{\alpha}{M^2 + \alpha} \leq 1 - \beta\alpha.
\end{aligned}
$$

Using (5.2.3), Assumptions 2.1.5 and 4.2.5 we have

$$
\begin{aligned}
u_{n+1,\alpha}^{h,\delta} - u_{\alpha}^{h,\delta} &= [I - \beta P_h(F'(u_{n,\alpha}^{h,\delta})^* F'(u_{n,\alpha}^{h,\delta}) + \alpha I](u_{n,\alpha}^{h,\delta} - u_{\alpha}^{h,\delta}) \\
&\quad -\beta[F'(u_{n,\alpha}^{h,\delta})^* \int_0^1 F'(u_{n,\alpha}^{h,\delta})\phi(u_{\alpha}^{h,\delta} + t(u_{n,\alpha}^{h,\delta} - u_{\alpha}^{h,\delta}), u_{n,\alpha}^{h,\delta}, u_{n,\alpha}^{h,\delta} - u_{\alpha}^{h,\delta})]dt \\
&\quad -\beta\phi_1(u_{n,\alpha}^{h,\delta}, u_{\alpha}^{h,\delta}, F'(u_{\alpha}^{h,\delta})^*(F(u_{\alpha}^{h,\delta}) - y^{\delta})) \\
&= [I - \beta(F'(u_{n,\alpha}^{h,\delta})^* F'(u_{n,\alpha}^{h,\delta}) + \alpha I)](u_{n,\alpha}^{h,\delta} - u_{\alpha}^{h,\delta}) \\
&\quad -\beta[F'(u_{n,\alpha}^{h,\delta})^* F'(u_{n,\alpha}^{h,\delta}) \int_0^1 \phi(u_{\alpha}^{h,\delta} + t(u_{n,\alpha}^{h,\delta} - u_{\alpha}^{h,\delta}), u_{n,\alpha}^{h,\delta}, u_{n,\alpha}^{h,\delta} - u_{\alpha}^{h,\delta})dt] \\
&\quad -\beta\phi_1(u_{n,\alpha}^{h,\delta}, u_{\alpha}^{h,\delta}, -\alpha(u_{\alpha}^{h,\delta} - u_0^{h,\delta})).
\end{aligned}
$$

Hence, using (5.2.4) we have

$$\begin{aligned}\|u_{n+1,\alpha}^{h,\delta}-u_\alpha^{h,\delta}\| &\leq (1-\alpha\beta)\|u_{n,\alpha}^{h,\delta}-u_\alpha^{h,\delta}\|+\beta M^2K_0\|u_{n,\alpha}^{h,\delta}-u_\alpha^{h,\delta}\|^2\int_0^1(1-t)dt \\ &\quad +\beta K_1\alpha\|u_{n,\alpha}^{h,\delta}-u_\alpha^{h,\delta}\|\|u_\alpha^{h,\delta}-u_0^{h,\delta}\| \\ &\leq (1-\alpha\beta+\alpha\beta K_1\|u_\alpha^{h,\delta}-u_0^{h,\delta}\|)\|u_{n,\alpha}^{h,\delta}-u_\alpha^{h,\delta}\|+\frac{\beta M^2K_0}{2}\|u_{n,\alpha}^{h,\delta}-u_\alpha^{h,\delta}\|^2 \\ &\leq (1-\alpha\beta+\alpha\beta K_1 r)\|u_{n,\alpha}^{h,\delta}-u_\alpha^{h,\delta}\|+\frac{\beta M^2K_0}{2}\|u_{n,\alpha}^{h,\delta}-u_\alpha^{h,\delta}\|^2.\end{aligned}$$

The above expression is of the form (4.2.5), where $a_n = 0, b = 1-\alpha\beta+\alpha\beta K_1 r$, $\gamma_n = \|x_{n,\alpha}^{h,\delta}-x_\alpha^{h,\delta}\|$ and $c = \frac{\beta M^2K_0}{2}$. We have by the condition on r, $b+2\sqrt{ac} = b < 1$ and

$$\gamma_0 = \|u_0^{h,\delta}-u_\alpha^{h,\delta}\| \leq \frac{1-b}{c} = \bar{\gamma}.$$

Hence by Lemma 4.2.2, we have

$$\begin{aligned}\|u_{n+1,\alpha}^{h,\delta}-u_\alpha^{h,\delta}\| &\leq (1-\alpha\beta+\alpha\beta K_1\|u_0^{h,\delta}-u_\alpha^{h,\delta}\|+\frac{\beta M^2K_0}{2}\|u_0^{h,\delta}-u_\alpha^{h,\delta}\|)\|u_{n,\alpha}^{h,\delta}-u_\alpha^{h,\delta}\| \\ &\leq (1-\alpha\beta+\alpha\beta K_1 r+\frac{\beta M^2K_0}{2}r)\|u_{n,\alpha}^{h,\delta}-u_\alpha^{h,\delta}\| \\ &\leq q_{\alpha,\beta}^h\|u_{n,\alpha}^{h,\delta}-u_\alpha^{h,\delta}\|. \qquad (5.2.5)\end{aligned}$$

Thus, since $r < \frac{2\alpha}{2\alpha K_1+M^2K_0}$, we have $q_{\alpha,\beta}^h < 1$, and hence

$$\|u_{n+1,\alpha}^{h,\delta}-u_\alpha^{h,\delta}\| < \|u_0^{h,\delta}-u_\alpha^{h,\delta}\| \leq r$$

and

$$\begin{aligned}\|u_{n+1,\alpha}^{h,\delta}-u_0\| &\leq \|u_{n+1,\alpha}^{h,\delta}-u_\alpha^{h,\delta}\|+\|u_\alpha^{h,\delta}-P_hu_0\|+\|(I-P_h)u_0\| \\ &\leq 2\|u_\alpha^{h,\delta}-P_hu_0\|+\|(I-P_h)u_0\| \\ &\leq 2\Big[\frac{\epsilon_h\|\hat{x}\|+\delta}{\sqrt{\alpha}}+r_0\Big]+c_0 < r.\end{aligned}$$

i.e., $u^{h,\delta}_{n+1,\alpha} \in B(u_0, r)$. Also, for $0 \le t \le 1$,

$$
\begin{aligned}
\|u^{h,\delta}_{\alpha} + t(u^{h,\delta}_{n+1,\alpha} - u^{h,\delta}_{\alpha}) - u_0\| &\le \|u^{h,\delta}_{\alpha} - u_0\| + \|u^{h,\delta}_{n+1,\alpha} - u^{h,\delta}_{\alpha}\| \\
&< \|u^{h,\delta}_{\alpha} - P_h u_0\| + \|(I - P_h)u_0\| + \|P_h u_0 - u^{h,\delta}_{\alpha}\| \\
&\le 2\|u^{h,\delta}_{\alpha} - P_h u_0\| + \|(I - P_h)u_0\| \le r.
\end{aligned}
$$

Hence, $u^{h,\delta}_{\alpha} + t(u^{h,\delta}_{n+1,\alpha} - u^{h,\delta}_{\alpha}) \in B(u_0, r)$ and A^h_{n+1} is well defined with $\|A^h_{n+1}\| \le M$. Thus, by induction $u^{h,\delta}_{n,\alpha}$ is well defined and remains in $B(u_0, r)$ for each $n = 0, 1, 2, \cdots$. By letting $n \to \infty$ in (5.1.3), we obtain the convergence of $u^{h,\delta}_{n,\alpha}$ to $u^{h,\delta}_{\alpha}$. The estimate (5.2.2) now follows from (5.2.5). □

For practical purpose we use the assumption ($\mathcal{A}_2$) to prove the convergence of $\{u^{h,\delta}_{n,\alpha}\}$ in (5.1.3) to $u^{h,\delta}_{\alpha}$.

THEOREM 5.2.3. *Let $u^{h,\delta}_{n,\alpha}$ be as in (5.1.3) and let $r < \frac{2\alpha}{L(2\sqrt{a_0}+M)}$. Then for each $\delta \in (0, \delta_0]$, $\alpha \in ((\delta + \epsilon_h)^2, a_0], \epsilon_h \le \epsilon_0$ the sequence $\{u^{h,\delta}_{n,\alpha}\}$ is in $B(u_0, r) \cap R(P_h)$ and converges to $u^{h,\delta}_{\alpha}$ as $n \to \infty$. Further*

$$
\|u^{h,\delta}_{n+1,\alpha} - u^{h,\delta}_{\alpha}\| \le (\bar{q}^h_{\alpha,\beta})^{n+1} \|P_h u_0 - u^{h,\delta}_{\alpha}\| \tag{5.2.6}
$$

where

$$
\bar{q}^h_{\alpha,\beta} = 1 - \alpha\beta + \beta L r \sqrt{a_0} + \frac{\beta M L}{2} r.
$$

Proof. From (5.2.3) and using Assumption ($\mathcal{A}_2$),

$$
\begin{aligned}
\|u^{h,\delta}_{n+1,\alpha} - u^{h,\delta}_{\alpha}\| &\le \|[I - \beta(P_h F'(u^{h,\delta}_{n,\alpha})^* F'(u^{h,\delta}_{n,\alpha}) P_h + \alpha I)]\| \|(u^{h,\delta}_{n,\alpha} - u^{h,\delta}_{\alpha})\| \\
&\quad + \beta \|P_h [F'(u^{h,\delta}_{n,\alpha})^* (A^h_n - F'(u^{h,\delta}_{n,\alpha}))]\| \|(u^{h,\delta}_{n,\alpha} - u^{h,\delta}_{\alpha})\| \\
&\quad + \beta \|P_h [F'(u^{h,\delta}_{\alpha})^* - F'(u^{h,\delta}_{n,\alpha})^*]\| \|(F(u^{h,\delta}_{\alpha}) - y^\delta)\| \\
&\le (1 - \alpha\beta)\|u^{h,\delta}_{n,\alpha} - u^{h,\delta}_{\alpha}\| + \frac{\beta M L}{2} \|u^{h,\delta}_{n,\alpha} - u^{h,\delta}_{\alpha}\|^2 \\
&\quad + \beta L \|u^{h,\delta}_{n,\alpha} - u^{h,\delta}_{\alpha}\| \|F(u^{h,\delta}_{\alpha}) - y^\delta\| \\
&\le (1 - \alpha\beta)\|u^{h,\delta}_{n,\alpha} - u^{h,\delta}_{\alpha}\| + \frac{\beta M L}{2} \|u^{h,\delta}_{n,\alpha} - u^{h,\delta}_{\alpha}\|^2 + \beta L r \sqrt{a_0} \|u^{h,\delta}_{n,\alpha} - u^{h,\delta}_{\alpha}\| \\
&\le (1 - \alpha\beta + \beta L \sqrt{a_0} r)\|u^{h,\delta}_{n,\alpha} - u^{h,\delta}_{\alpha}\| + \frac{\beta M L}{2} \|u^{h,\delta}_{n,\alpha} - u^{h,\delta}_{\alpha}\|^2.
\end{aligned}
$$

The rest of the proof is analogous to the proof of Theorem 5.2.2. □

5.3 ERROR BOUNDS UNDER SOURCE CONDITIONS

We use estimate in Theorem 5.2.2 for obtaining error estimate for $\|u_{n,\alpha}^{h,\delta} - \hat{x}\|$. Similar result can be obtained using the estimate in Theorem 5.2.3. Note that we have

$$P_hF'(u_\alpha^{h,\delta})^*(F(u_\alpha^{h,\delta}) - y^\delta) + \alpha P_h(u_\alpha^{h,\delta} - u_0) = 0 \qquad (5.3.1)$$

and

$$P_hF'(u_\alpha^{\delta})^*(F(u_\alpha^{\delta}) - y^\delta) + \alpha P_h(u_\alpha^{\delta} - u_0) = 0. \qquad (5.3.2)$$

So by (5.3.1) and (5.3.2) we obtain

$$\begin{aligned} &P_hF'(u_\alpha^{h,\delta})^*F(u_\alpha^{h,\delta}) - P_hF'(u_\alpha^{\delta})^*F(u_\alpha^{\delta}) - [P_hF'(u_\alpha^{h,\delta})^* - P_hF'(u_\alpha^{\delta})^*]y^\delta \\ &+\alpha P_h(u_\alpha^{h,\delta} - u_\alpha^\delta) = 0. \end{aligned}$$

That is

$$\begin{aligned} & (P_hF'(u_\alpha^{h,\delta})^*F'(u_\alpha^{h,\delta})P_h + \alpha I)(u_\alpha^{h,\delta} - P_h u_\alpha^\delta) \\ = \;& P_hF'(u_\alpha^{h,\delta})^*[F'(u_\alpha^{h,\delta}) - T](u_\alpha^{h,\delta} - u_\alpha^\delta) \\ & +P_hF'(u_\alpha^{h,\delta})^*F'(u_\alpha^{h,\delta})(I - P_h)u_\alpha^\delta \\ & +(P_hF'(u_\alpha^{\delta})^* - P_hF'(u_\alpha^{h,\delta})^*)(F(u_\alpha^\delta) - y^\delta) \end{aligned}$$

where $T = \int_0^1 F'(u_\alpha^\delta + t(u_\alpha^{h,\delta} - u_\alpha^\delta))dt$. So by Assumption 2.1.5 and 4.2.5 we have

$$\begin{aligned} & \|u_\alpha^{h,\delta} - P_h u_\alpha^\delta\| \\ = \;& \|[P_hF'(u_\alpha^{h,\delta})^*F'(u_\alpha^{h,\delta})P_h + \alpha I]^{-1} \times [P_hF'(u_\alpha^{h,\delta})^*[T - F'(u_\alpha^{h,\delta})](u_\alpha^{h,\delta} - u_\alpha^\delta)) \\ & +P_hF'(u_\alpha^{h,\delta})^*F'(u_\alpha^{h,\delta})(I - P_h)u_\alpha^\delta + P_h(F'(u_\alpha^{\delta})^* - F'(u_\alpha^{h,\delta})^*)(F(u_\alpha^\delta) - y^\delta)]\| \\ \leq \;& \|(P_hF'(u_\alpha^{h,\delta})^*F'(u_\alpha^{h,\delta})P_h + \alpha I)^{-1}P_hF'(u_\alpha^{h,\delta})^*F'(u_\alpha^{h,\delta})[P_h + I - P_h] \\ & \times \int_0^1 \phi(u_\alpha^\delta + t(u_\alpha^{h,\delta} - u_\alpha^\delta), u_\alpha^{h,\delta}, u_\alpha^{h,\delta} - u_\alpha^\delta)dt\| + \frac{\epsilon_h}{\sqrt{\alpha}}\|u_\alpha^\delta\| \\ & +\frac{1}{\alpha}\|\phi_1(u_\alpha^\delta, u_\alpha^{h,\delta}, F'(u_\alpha^\delta)^*(F(u_\alpha^\delta) - y^\delta))\| \end{aligned}$$

$$
\begin{aligned}
&\leq K_0\big(1+\frac{\epsilon_h}{\sqrt{\alpha}}\big)\int_0^1(1-t)dt\|u_\alpha^{h,\delta}-u_\alpha^\delta\|\|u_\alpha^{h,\delta}-u_\alpha^\delta\|+\frac{\epsilon_h}{\sqrt{\alpha}}\|u_\alpha^\delta\| \\
&\quad +\frac{K_1}{\alpha}\|u_\alpha^h-u_\alpha^{h,\delta}\|\|\alpha(u_\alpha^\delta-u_0)\| \\
&\leq \frac{K_0}{2}\big(1+\frac{\epsilon_h}{\sqrt{\alpha}}\big)2r\|u_\alpha^{h,\delta}-u_\alpha^\delta\|+\frac{\epsilon_h}{\sqrt{\alpha}}\|u_\alpha^\delta\|+K_1r\|u_\alpha^\delta-u_\alpha^{h,\delta}\| \\
&\leq 2(K_0+K_1)r(\|u_\alpha^{h,\delta}-P_hu_\alpha^\delta\|+\|P_hu_\alpha^\delta-u_\alpha^\delta\|)+\frac{\epsilon_h}{\sqrt{\alpha}}\|u_\alpha^\delta\|
\end{aligned}
$$

hence

$$
\begin{aligned}
\|u_\alpha^{h,\delta}-P_hu_\alpha^\delta\| &\leq \frac{1}{1-2(K_0+K_1)r}\big[2(K_0+K_1)r\|(I-P_h)u_\alpha^\delta\| \\
&\quad +\frac{\epsilon_h}{\sqrt{\alpha}}(r+\|u_0\|)\big]. \qquad (5.3.3)
\end{aligned}
$$

THEOREM 5.3.1. *Let the assumptions in Theorem 5.2.2 holds and let $u_{n,\alpha}^{h,\delta}$ be as in (5.1.3). Then*

$$
\begin{aligned}
\|u_{n,\alpha}^{h,\delta}-\hat{x}\| &\leq (q_{\alpha,\beta}^h)^nr+\frac{1}{1-2(K_0+K_1)r}\big(b_h+\frac{\epsilon_h}{\sqrt{\alpha}}(r+\|u_0\|)\big) \\
&\quad +\bar{K}(\frac{\delta}{\sqrt{\alpha}}+\Phi(\alpha))
\end{aligned}
$$

where $\bar{K}=\frac{2(1-(K_0+K_1)r)}{1-2(K_0+K_1)r}\times\frac{1}{1+K_0+K_1-\frac{r}{2}(2K_0+K_1)}$.

Proof. By triangle inequality, we have $\|u_{n,\alpha}^{h,\delta}-\hat{x}\|\leq\|u_{n,\alpha}^{h,\delta}-u_\alpha^{h,\delta}\|+\|u_\alpha^{h,\delta}-u_\alpha^\delta\|+\|u_\alpha^\delta-\hat{x}\|$. Therefore by (5.3.3), Theorem 4.3.2 and Theorem 5.2.2 , we obtain that

$$
\begin{aligned}
\|u_{n,\alpha}^{h,\delta}-\hat{x}\| &\leq (q_{\alpha,\beta}^h)^nr+\|u_\alpha^{h,\delta}-P_hu_\alpha^\delta\|+\|(I-P_h)u_\alpha^\delta\|+\|u_\alpha^\delta-\hat{x}\| \\
&\leq (q_{\alpha,\beta}^h)^nr+\frac{1}{1-2(K_0+K_1)r}[2(K_0+K_1)r\|(I-P_h)u_\alpha^\delta\|+\frac{\epsilon_h}{\sqrt{\alpha}}(r+\|u_0\|)] \\
&\quad \|(I-P_h)u_\alpha^\delta\|+\|u_\alpha^\delta-\hat{x}\| \\
&\leq (q_{\alpha,\beta}^h)^nr+\frac{1}{1-2(K_0+K_1)r}\|(I-P_h)u_\alpha^\delta\|
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{1}{1-2(K_0+K_1)r}\frac{\epsilon_h}{\sqrt{\alpha}}(r+\|u_0\|)+\|u_\alpha^\delta-\hat{x}\| \\
\leq\ & (q_{\alpha\beta}^h)^n r+\frac{1}{1-2(K_0+K_1)r}[\|(I-P_h)(u_\alpha^\delta-\hat{x})\|+\|(I-P_h)\hat{x}\|] \\
& +\frac{1}{1-2(K_0+K_1)r}\frac{\epsilon_h}{\sqrt{\alpha}}(r+\|u_0\|)+\|u_\alpha^\delta-\hat{x}\| \\
\leq\ & (q_{\alpha,\beta}^h)^n r+\frac{1}{1-2(K_0+K_1)r}[b_h+\frac{\epsilon_h}{\sqrt{\alpha}}(r+\|u_0\|)] \\
& +\Big(\frac{1}{1-2(K_0+K_1)r}+1\Big)\|u_\alpha^\delta-\hat{x}\| \\
\leq\ & (q_{\alpha,\beta}^h)^n r+\frac{1}{1-2(K_0+K_1)r}\big[b_h+\frac{\epsilon_h}{\sqrt{\alpha}}(r+\|u_0\|)\big] \\
& +\frac{2(1-(K_0+K_1)r)}{1-2(K_0+K_1)r}\|u_\alpha^\delta-\hat{x}\|.
\end{aligned}
$$

The result now follows from Theorem 4.3.2. □

Let

$$n_\delta^h=\min\{n:q_{\alpha,\beta}^n\leq\frac{\delta+\epsilon_h}{\sqrt{\alpha}}\}\text{ and }b_h\leq\frac{\delta+\epsilon_h}{\sqrt{\alpha}}. \tag{5.3.4}$$

Then by Theorem 5.3.1, we have

$$\|u_{n_\delta^h,\alpha}^{h,\delta}-\hat{x}\|\leq C\big(\frac{\delta+\epsilon_h}{\sqrt{\alpha}}+\Phi(\alpha)\big) \tag{5.3.5}$$

where $C=r+\frac{1+r+\|x_0\|}{1-2(K_0+K_1)r}+\bar{K}$.

5.3.1 A priori choice of the parameter

Let $\psi:(0,\|F'(\hat{x})\|^2]\to[0,\infty)$ be defined by $\psi(\lambda):=\lambda\sqrt{\Phi^{-1}(\lambda)}$. Then as in the Chapter 4, one can see that for $\alpha_\delta=\Phi^{-1}(\psi^{-1}(\delta+\epsilon_h))$ we obtain the optimal order error estimate. In fact we have the following theorem.

THEOREM 5.3.2. *Let $\psi(\lambda):=\lambda\sqrt{\Phi^{-1}(\lambda)}$ for $0<\lambda\leq\|F'(\hat{x})\|^2$, and let the assumptions in Theorem 5.3.1 holds. For $\delta\in(0,\delta_0]$, let $\alpha:=\alpha_\delta=$*

$\Phi^{-1}(\psi^{-1}(\delta + \epsilon_h))$ *and let* n_δ *be as in (5.3.4). Then*

$$\|x^{\delta}_{n^h_\delta,\alpha} - \hat{x}\| = O(\psi^{-1}(\delta + \epsilon_h)).$$

5.3.2 Balancing Principle: for FDTGTM

Let

$$n_i := \min\{n : q^h_{\alpha,\beta} \leq \frac{\delta + \epsilon_h}{\sqrt{\alpha_i}}\}$$

and let $u^{h,\delta}_{\alpha_i} := u^{h,\delta}_{n_i,\alpha_i}$. Then from (5.3.5), we have

$$\|u^{h,\delta}_{\alpha_i} - \hat{x}\| \leq C(\frac{\delta + \epsilon_h}{\sqrt{\alpha_i}} + \Phi(\alpha_i)), \forall i = 1, 2, ..., N.$$

As in the Section 4.3.2 we consider the balancing principle suggested by Pereverzev and Schock (2005), for choosing the regularization parameter α from the set D_N defined by

$$D_N := \{\alpha_i = \mu^{2i}\alpha_0 < 1, i = 1, 2, ..., N\},$$

where $\alpha_0 = (\delta + \epsilon_h)^2$ (see Semenova (2010)) and $\mu > 1$.

The main result of this section is the following theorem, proof of which is analogous to the proof of Theorem 4.4 in George and Nair (2008).

THEOREM 5.3.3. *Assume that there exists* $i \in \{0, 1, ..., N\}$ *such that* $\Phi(\alpha_i) \leq \frac{\delta+\epsilon_h}{\sqrt{\alpha_i}}$. *Let assumptions of Theorem 5.3.1 be satisfied and let*

$$l := \max\left\{i : \Phi(\alpha_i) \leq \frac{\delta + \epsilon_h}{\sqrt{\alpha_i}}\right\} < N,$$

$$k := \max\left\{i : \forall j = 1, 2, ..., i-1; \|u^{\delta}_{\alpha_i} - u^{h,\delta}_{\alpha_j}\| \leq 4C\frac{\delta + \epsilon_h}{\sqrt{\alpha_j}}\right\}$$

where C *is as in (5.3.5). Then* $l \leq k$ *and*

$$\|u^{h,\delta}_{\alpha_i} - \hat{x}\| \leq 6C\mu\psi^{-1}(\delta + \epsilon_h).$$

As per Theorem 5.3.3, the choice of the regularization parameter involves the following steps:

- Choose $\alpha_0 = (\delta + \epsilon_h)^2$
- Choose $\alpha_i := \mu^{2i}\alpha_0$, $i = 0, 1, 2, \cdots, N$ with and $\mu > 1$.

Algorithm

1. Set $i = 0$.
2. Choose $n_i := \min\left\{n : q^h_{\alpha_i,\beta} \leq \frac{\delta+\epsilon_h}{\sqrt{\alpha_i}}\right\}$.
3. Solve $u_i^{h,\delta} := u_{n_i,\alpha_i}^{h,\delta}$ by using the iteration (5.1.3).
4. If $\|u_i^{h,\delta} - u_j^{h,\delta}\| > 4C\frac{1}{\mu^j}, j < i$, then take $k = i - 1$ and return u_k.
5. Else set $i = i + 1$ and go to 2.

5.4 IMPLEMENTATION OF THE METHOD

Let X_M be a sequence of finite dimensional subspaces of X and let $P_h, (h = \frac{1}{M})$ denote the orthogonal projection on X with $R(P_h) = X_M$. Let $\{\Phi_1, \Phi_2, \ldots, \Phi_M\}$ be a basis for X_M. We assume that $\|P_h x - x\| \to 0$ as $h \to 0$ $\forall x \in X$.

Since $u_{n,\alpha}^{h,\delta} \in X_M, u_{n,\alpha}^{h,\delta} = \sum_{i=1}^M \lambda_i^n \Phi_i$ for some $\lambda_1^n, \lambda_2^n, ..., \lambda_M^n \in \mathbb{R}$. Then from (5.1.3) we have,

$$\begin{aligned}\sum_{i=1}^M (\lambda_i^{n+1} - \lambda_i^n)\Phi_i &= \beta P_h F'(u_{n,\alpha}^{h,\delta})^* \Big(\sum_{i=1}^M (\eta_i - F_i)\Phi_i\Big) \\ &\quad + \alpha\beta \sum_{i=1}^M (X_{0,i} - \lambda_i^n)\Phi_i,\end{aligned}$$

where $P_h[y^\delta - F(u_{n,\alpha}^{h,\delta})] = \sum_{i=1}^M (\eta_i - F_i)\Phi_i$ and $P_h(u_0^{h,\delta} - u_{n,\alpha}^{h,\delta}) = \sum_{i=1}^M (X_{0,i} - \lambda_i^n)\Phi_i, i = 1, 2, \cdots M$. Then $u_{n+1,\alpha}^{h,\delta} - u_{n,\alpha}^{h,\delta}$ is a solution of (5.1.3) if and only if $[\lambda^{n+1} - \lambda^n] = [\lambda_1^{n+1} - \lambda_1^n, \lambda_2^{n+1} - \lambda_2^n, ..., \lambda_M^{n+1} - \lambda_M^n]^T$ is the unique solution of

$$B_M[\lambda^{n+1} - \lambda^n] = \beta M_M[\eta - F^M] + \alpha\beta B_M[X_0 - \lambda^M],$$

where $M_M = (\langle \Phi_i, F'(x_{n,\alpha}^{h,\delta})\Phi_j \rangle), B_M = (\langle \Phi_i, \Phi_j \rangle) \;\; i,j = 1,2,...,M,$

$$F^M = [F_1, F_2, \cdots, F_M]^T,\ \eta = [\eta_1, \eta_2, \cdots, \eta_M]^T,$$

$$X_0 = [X_{0,1}, X_{0,2}, \cdots, X_{0,M}]^T \text{ and } \lambda^M = [\lambda_1^n, \lambda_2^n, \cdots, \lambda_M^n]^T.$$

5.5 NUMERICAL EXAMPLES

In this section first we consider Example 3.6.1 considered in Chapter 3 and we compare the experimental results of the proposed method with that of method (3.2.1).

EXAMPLE 5.5.1. *Returning back to Example 3.6.1. We use the same basis used in Example 3.6.1. The results of numerical experiments for method (5.1.3) and (3.2.1) for different values of δ and m are presented in Table 5.1*

Table 5.1: Comparison table for relative and residual error of the example 3.6.1

Methods		Method (5.1.3)			Method (3.2.1)		
m	$\delta + \epsilon_h$	α_k	Δ_1	Δ_2	α_k	Δ_1	Δ_2
35	0.1	0.0506	1.8577×10^{-4}	3.7391×10^{-5}	0.3240	0.2281	1.9605
	0.02	0.0020	1.8576×10^{-4}	3.7982×10^{-5}	0.0648	0.2280	1.9606
	0.01	0.0005	1.8576×10^{-4}	3.8057×10^{-5}	0.0324	0.2277	1.9606
40	0.1	0.0506	1.5491×10^{-4}	2.8882×10^{-5}	0.3240	0.2468	1.9598
	0.02	0.0020	1.5491×10^{-4}	2.9269×10^{-5}	0.0648	0.2484	1.9598
	0.01	0.0005	1.5490×10^{-4}	2.9317×10^{-5}	0.0324	0.2481	1.9599

Comparison Table 5.1 shows that the relative and residual error for method (5.1.3) is smaller than that of method (3.2.1).

The plots in the Figure 5.1 to 5.8 gives the exact and approximate solution for different values of m and δ by using the method (5.1.3).

Next, example we consider an integral equation considered in Semenova (2010).

Figure 5.1: Exact solution for $m = 35$

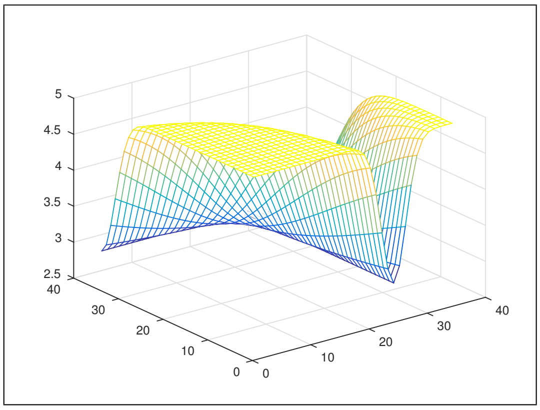


Figure 5.2: Approximate solution for $m = 35$ and $\delta = 0.1$

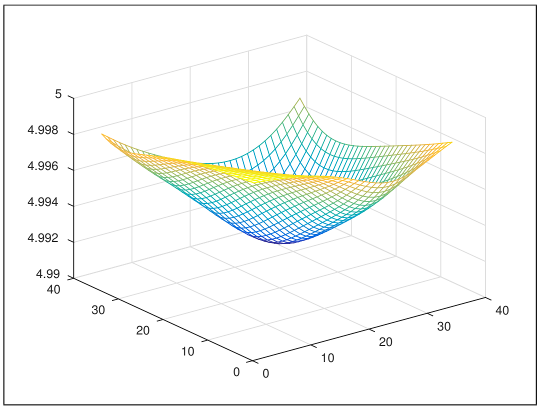


Figure 5.3: Approximate solution for $m = 35$ and $\delta = 0.02$

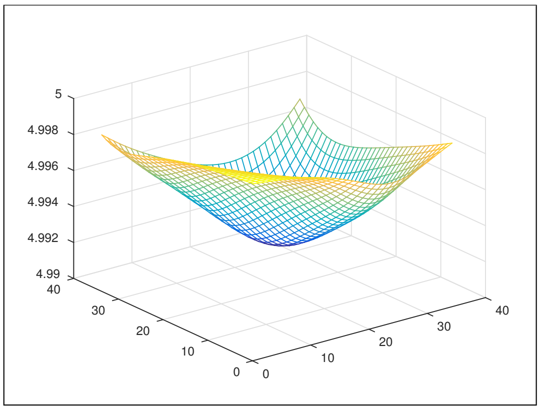


Figure 5.4: Approximate solution for $m = 35$ and $\delta = 0.01$

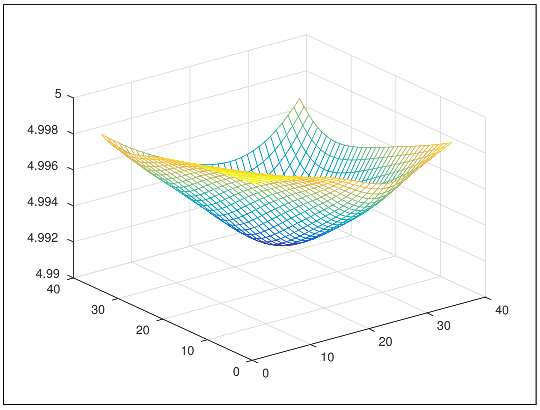


Figure 5.5: Exact solution for $m = 40$

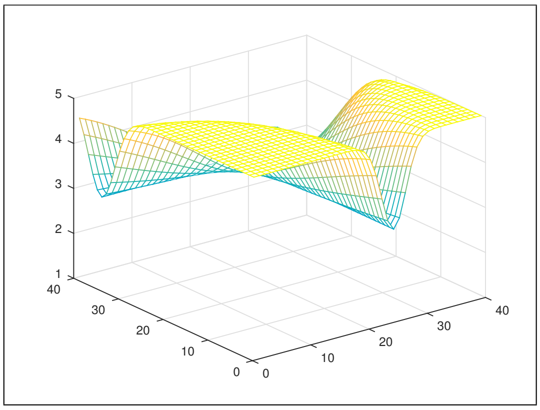


Figure 5.6: Approximate solution for $m = 40$ and $\delta = 0.1$

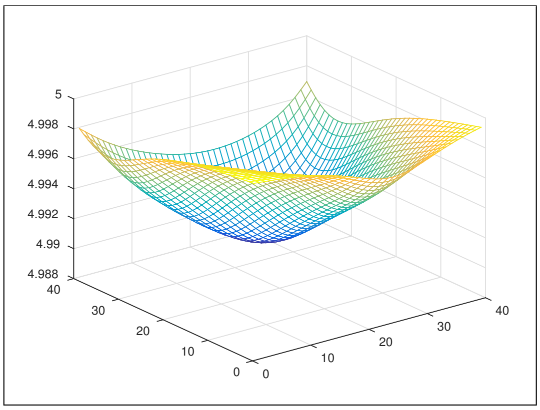


Figure 5.7: Approximate solution for $m = 40$ and $\delta = 0.02$

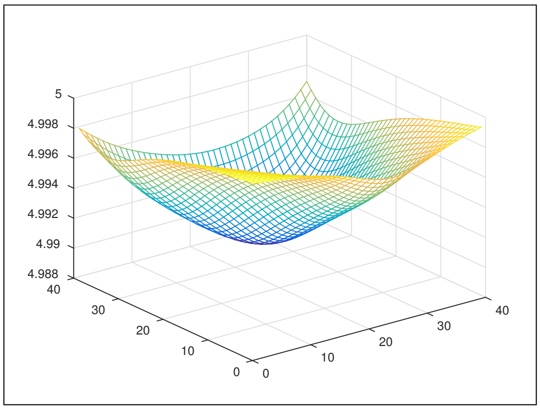
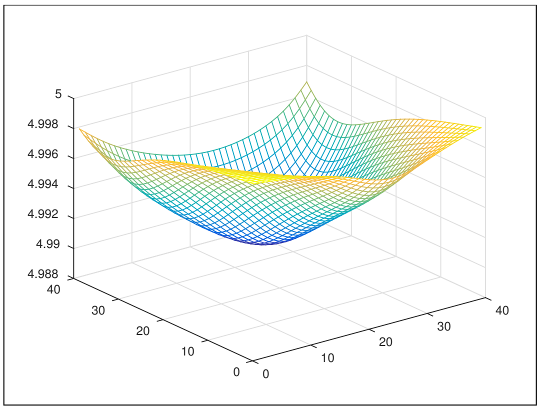


Figure 5.8: Approximate solution for $m = 40$ and $\delta = 0.01$



EXAMPLE 5.5.2. *(Semenova (2010), section 4.3) Let* $F : D(F) \subseteq H^1(0,1) \to L^2(0,1)$ *defined by*

$$F(u) = \int_0^1 k(s,t)u^3(s)ds,$$

where

$$k(s,t) = \begin{cases} (1-t)s, & 0 \le s \le t \le 1 \\ (1-s)t, & 0 \le t \le s \le 1 \end{cases}.$$

The Fréchet derivative of F *is given by*

$$F'(u)w = 3\int_0^1 k(s,t)u^2(s)w(s)ds.$$

We take

$$f(t) = \frac{6\sin(\pi t) + \sin^3(\pi t)}{9\pi^2}$$

and $f^\delta = f + \delta$. *Then the exact solution*

$$\hat{x}(t) = \sin(\pi t).$$

We use

$$x_0(t) = \sin(\pi t) + \frac{3[t\pi^2 - t^2\pi^2 + \sin^2(\pi t)]}{4\pi^2}$$

as our initial guess.

We apply algorithm by choosing a sequence of finite dimensional subspace (X_M) *of* X *with dim* $X_M = M + 1$. *Precisely we choose* X_M *as the linear span of* $\{\Phi_1, \Phi_2, \Phi_3, ..., \Phi_{M+1}\}$ *where* $\Phi_i, i = 1, 2, ..., M + 1$ *are linear splines in a uniform grid of* $M + 1$ *points in* $[0, 1]$.

In Table 5.2 the results of numerical experiments for different values of δ *and* M *are presented. Here* Δ_1 and Δ_2 *are as in Example 3.6.1.*

Table 5.2: Comparison table for relative and residual error of the example 5.5.2

Methods		Method (5.1.3)			Method (3.2.1)		
M	$\delta + \epsilon_h$	α_k	Δ_1	Δ_2	α_k	Δ_1	Δ_2
32	0.0133	0.2561	0.0106	0.0027	0.0178	0.0286	0.2420
64	0.0133	0.2560	0.0076	0.0012	0.0177	0.0239	0.1365
128	0.0133	0.2559	0.0054	0.0006	0.0177	0.0122	0.0730
256	0.0133	0.2559	0.0038	0.0003	0.2559	0.0366	0.0378
512	0.0133	0.2559	0.0027	0.0001	0.2559	0.0262	0.0192
1024	0.0133	0.2559	0.0019	0.00006	0.2559	0.0187	0.0097

Comparison Table 5.2 shows that the relative and residual error for method (5.1.3) is smaller than that of method (3.2.1).

The plots in the Figure 5.9 to 5.14 gives the exact and approximate solution for different values of m *and by using the method (5.1.3).*

Figure 5.9: Curves of the exact and approximate solutions when $M = 32$

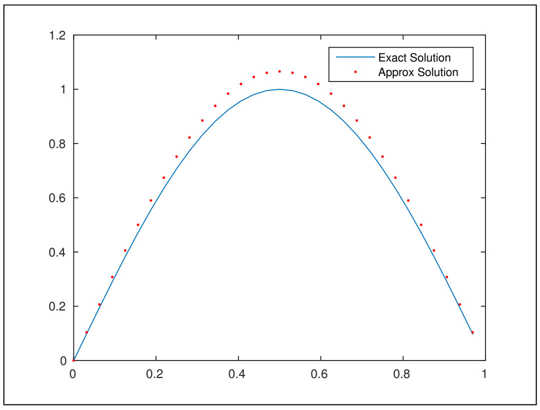


Figure 5.10: Curves of the exact and approximate solutions when $M = 64$

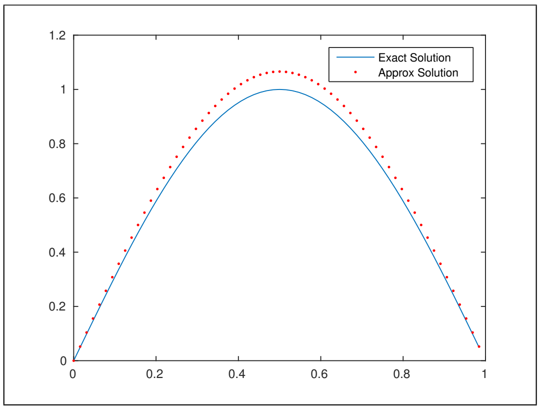


Figure 5.11: Curves of the exact and approximate solutions when $M = 128$

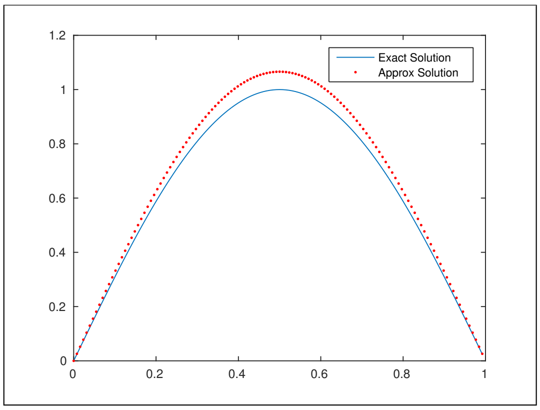


Figure 5.12: Curves of the exact and approximate solutions when $M = 256$

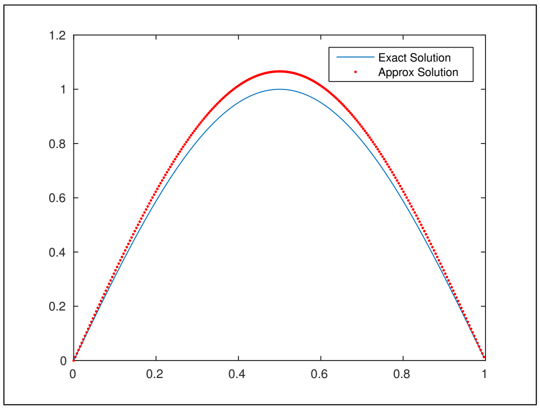


Figure 5.13: Curves of the exact and approximate solutions when $M = 512$

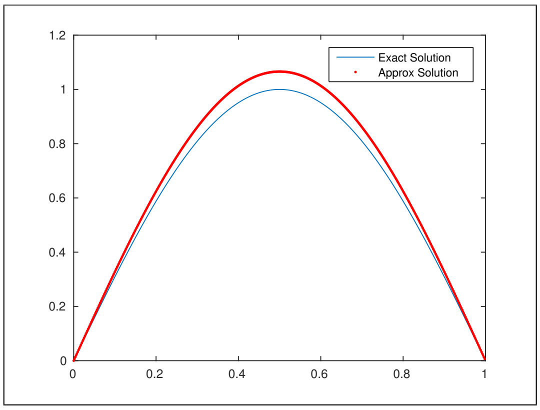
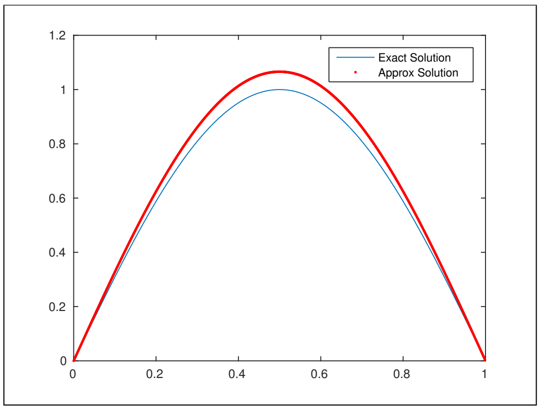


Figure 5.14: Curves of the exact and approximate solutions when $M = 1024$



5.6 CONCLUSION

In this chapter, we considered the finite dimensional realization of the method considered in Chapter 4. The convergence of the method in finite dimensional space is provided in this chapter. We provided two numerical examples and compared the numerical results with that of the method considered in Chapter 3.

Chapter 6

CONCLUSION

In this thesis we have considered the problem of approximately solving nonlinear ill-posed equation

$$F(x) = y. \tag{6.0.1}$$

We have used iterative Lavrentiev regularization method when $F : D(F) \subseteq X \to X$ is monotone and iterative Thikhnov type regularization is used when $F : D(F) \subseteq X \to Y$ where X and Y are Hilbert spaces. Regularization parameter α is chosen according to the adaptive method considered by Pereverzev and Schock (2005) for the linear operator equations.

Throughout this thesis we assume that the available data is y^δ with $\|y - y^\delta\| \leq \delta$.

In Chapter 1 we considered some basic definitions and results.

In Chapter 2 we considered the problem of approximately solving (6.0.1) when $F : D(F) \subseteq X \to X$ is monotone operator. Here we used iterative Lavrentiev regularization method to obtain the approximate solution for the operator equation (6.0.1).

The finite dimensional realization of the method considered in Chapter 2 is considered in Chapter 3.

In Chapter 4 we considered (6.0.1) when $F : D(F) \subseteq X \to Y$ is weakly closed, continuous and convex in the domain of F. We used iterative Thikhnov type regularization to get approximate solution.

Chapter 5 gives the finite dimensional realization of the method considered in Chapter 4 and numerical example provides the efficiency of the method considered.

The theory of non-linear ill-posed problems are well developed and can be considered as almost complete in Hilbert space. Hence in future we deal with non linear ill-posed problems in Banach space.

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