ON THE SOLUTIONS OF VISCOUS BURGERS EQUATIONS

Thesis

Submitted in partial fulfillment of the requirements for the degree of

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by

MOHD AHMED

DEPARTMENT OF MATHEMATICAL & COMPUTATIONAL SCIENCES

NATIONAL INSTITUTE OF TECHNOLOGY KARNATAKA

SURATHKAL, MANGALORE - 575025

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To my family.

DECLARATION

By the Ph.D. Research Scholar

I hereby **declare** that the thesis entitled **"ON THE SOLUTIONS OF VISCOUS BURGERS EQUATIONS"** which is being submitted to the **National Institute of Technology Karnataka, Surathkal** in partial fulfillment of the requirements for the award of the degree of **Doctor of Philosophy** in **Department of Mathematical and Computational Sciences** is a **bonafide report of the research work carried out by me**. The material contained in this thesis has not been submitted to any University or Institution for the award of any degree.

Place : NITK, Surathkal.

Date :13-11-2017.

Mohd Ahmed

Reg. No. 138035 MA13F06,

Department of MACS,

NITK, Surathkal.

CERTIFICATE

This is to **certify** that the thesis entitled **"ON THE SOLUTIONS OF VISCOUS BURGERS EQUATIONS"** submitted by **Mohd Ahmed**, (Reg. No. 138035 MA13F06) as the record of the research work carried out by him, is *accepted as the thesis submission* in partial fulfillment of the requirements for the award of degree of **Doctor of Philosophy**.

Dr. Satyanarayana E.
Guide

Dr. Murugan V.
Co-Guide

Chairman - DRPC

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ABSTRACT

The viscous Burgers equation $u_t + uu_x = \nu u_{xx}$, is a nonlinear partial differential equation, named after the great physicist *Johannes Martinus Burgers* (1895-1981). This equation can be linearized to the heat equation through Cole-Hopf transformation. First, we study asymptotic behavior of solutions to an initial value problem posed for heat equation. For which, we construct an approximate solution to the initial value problem in terms of derivatives of Gaussian by incorporating the moments of initial function. Spatial shifts are introduced into the leading order term as well as penultimate term of the approximation. We extend these results to observe asymptotic behavior of solutions to the viscous Burgers equation.

Secondly, we deal with a forced Burgers equation (FBE) subject to the initial function, which is continuous and summable on $\mathbb{R}$. Large time asymptotic behavior of solutions to the FBE is determined with precise error estimates. To achieve this, we construct solutions for the FBE with a different initial class of functions using the method of separation of variables and Cole-Hopf like transformation. These solutions are constructed in terms of Hermite polynomials with the help of similarity variables. The constructed solutions would help us to pick up an asymptotic approximation and to show that the magnitude of the difference function of the true and approximate solutions decays algebraically to zero for large time.

Keywords : Diffusion equation; Burgers equation ; Forced Burgers equation ; Cole-Hopf transformation ; Large time asymptotics.

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Chapter 1

General Introduction

The viscous Burgers equation is a second order, non-linear and parabolic partial differential equation of the form

$$u_t + uu_x = \nu u_{xx}, \quad x \in \mathbb{R}, \ t > 0, \tag{1.0.1}$$

where $\nu > 0$ is the coefficient of viscosity and $u := u(x,t)$ is the velocity of fluid. This equation was named after the great physicist *Johannes Martinus Burgers* (1895-1981). Bateman (1915) first introduced the equation (1.0.1) in a physical context. Subsequently, Burgers (1948) studied this equation primarily to shed light on the study of turbulence described by the interaction of two opposite effects of nonlinear convection and linear diffusion. Equation (1.0.1) received considerable study because it is one of the simplest examples of nonlinear partial differential equations having diffusion and nonlinear convection terms. If the coefficient of viscosity $\nu = 0$, then (1.0.1) becomes

$$u_t + uu_x = 0, \quad x \in \mathbb{R}, \ t > 0, \tag{1.0.2}$$

which is a first order hyperbolic partial differential equation and is known as the inviscid Burgers equation.

Burgers equation arises in many physical applications such as modeling traffic flow, fluid flow in certain conditions, magneto-hydrodynamics, sound waves in a viscous medium, shock waves in a viscous medium and waves in fluid-filled viscous elastic tubes (cf.(Whitham, 1974)). Due to huge applications in various fields, this equation has attracted several researchers, both mathematicians and engineers to study its properties by using various analytical as well as numerical techniques. This equation is also used for testing and comparing the accuracy of numerical techniques as it has solutions for a huge class of initial data (see (Fletcher, 1982)). Benton and Platzman (1972) listed out 35 distinct solutions to the initial value problem for Burgers equation in the infinite domain as well as two other solutions for the initial and boundary value problem in the finite domain.

1.1 Two simple applications of Burgers equation

Burgers equation often appears as a simplification of more complex and sophisticated model. Hence it is thought as a tool to understand some of the inside behavior of the general problem. Here we will present two cases, where Burgers equation appears as a model equation.

1.1.1 Simplification of Navier-Stokes equations

Consider the Navier-Stokes equations

$$\begin{cases} \nabla . v = 0, \\ \rho v_t + \rho v \cdot (\nabla v) + \nabla p - \nu \nabla^2 v = 0. \end{cases} \tag{1.1.3}$$

It is well known that, if ρ is considered to be the density, p the pressure, v the velocity and ν the viscosity of a fluid, then (1.1.3) describe the dynamics of a divergence free, incompressible ($\rho_t = 0$) flow where gravitational effects are negligible.

Rewriting the second equation in (1.1.3) for the x-component of velocity vector, which we call v^x, we get

$$\rho \frac{\partial v^x}{\partial t} + \rho v^x \frac{\partial v^x}{\partial x} + \rho v^y \frac{\partial v^x}{\partial y} + \rho v^z \frac{\partial v^x}{\partial z} + \frac{\partial p}{\partial x} - \nu \left(\frac{\partial^2 v^x}{\partial x^2} + \frac{\partial^2 v^x}{\partial y^2} + \frac{\partial^2 v^x}{\partial z^2} \right) = 0. \tag{1.1.4}$$

If we consider a one-dimensional problem with no pressure gradient, then the above equation reduces to

$$\rho \frac{\partial v^x}{\partial t} + \rho v^x \frac{\partial v^x}{\partial x} - \nu \left(\frac{\partial^2 v^x}{\partial x^2} \right) = 0.$$

If we use u in place of v^x and take ϵ to be kinematic viscosity, i.e $\epsilon = \frac{\nu}{\rho}$, then the last equation becomes the viscous Burgers equation as it has been presented in (1.0.1), i.e.,

$$u_t + uu_x = \epsilon u_{xx}, \quad x \in \mathbb{R}, \ t > 0.$$

For further details one can see (Ames, 1965).

1.1.2 Traffic flow problem

Consider a street starting at a point x_1 and ending at a point x_2 . Let $u(x,t)$ be the density of cars at a point $x \in [x_1, x_2]$ and at time $t > 0$. Therefore, the total number of cars between points x_1 and x_2 at time t can be represented by

$$\int_{x_1}^{x_2} u(x,t)dx. \tag{1.1.5}$$

Now the rate of change in the number of cars between points x_1 and x_2 at time t is given by

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)dx = f(u(x_1,t)) - f(u(x_2,t)), \tag{1.1.6}$$

where f represents the flow rate onto and off the street. Assuming u and f to be continuously differentiable functions, we see that

$$\int_{x_1}^{x_2} u_t(x,t)dx = -\int_{x_1}^{x_2} (f(u))_x dx. \tag{1.1.7}$$

Therefore, we can say that the density of cars at point x and at time t satisfies the partial differential equation

$$u_t + (f(u))_x = 0. \tag{1.1.8}$$

If the flow rate is defined by the function $f(u) = \frac{u^2}{2}$, then we get

$$u_t + uu_x = 0, \tag{1.1.9}$$

which is the inviscid Burgers equation. For further details one can see (Knobel, 2000).

1.2 Cauchy problem for the viscous Burgers equation

The Cauchy problem or the initial value problem for the viscous Burgers equation is given by

$$u_t + uu_x = \nu u_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \tag{1.2.10}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{1.2.11}$$

where u_0 is a given function. It is known that the Cauchy problem (1.2.10)-(1.2.11) is reduced to the Cauchy problem for the heat equation

$$\phi_t = \nu\phi_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \tag{1.2.12}$$

$$\phi(x,0) = \exp\left\{-\frac{1}{2\nu}\int_0^x u_0(y)dy\right\} =: \phi_0(x), \quad x \in \mathbb{R}, \tag{1.2.13}$$

by using Cole-Hopf transformation discovered independently by Hopf (1950) and Cole (1951), which is

$$\phi(x,t) = \exp\left\{-\frac{1}{2\nu}\int_0^x u(y,t)dy\right\}. \tag{1.2.14}$$

It is to be noted that one may even take the lower limit of the integral in right hand side of (1.2.14) to be any real number or $-\infty$. It is well known that the solution for the Cauchy problem (1.2.12)-(1.2.13) is given by (see Theorem 1.3.1)

$$\phi(x,t) = \frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\nu t}}\phi_0(y)dy, \tag{1.2.15}$$

for the initial data satisfying either $\phi_0 \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ or $\phi_0 \in L^1(\mathbb{R})$. Assuming that the initial data $u_0(x)$ is integrable in every finite x-interval and

$$\int_0^x u_0(y)dy = o(x^2), \quad \text{for large } |x|,$$

Hopf (1950) derived explicit solution $u(x,t)$ as follows:

$$\begin{aligned} u(x,t) &= -2\nu\frac{\phi_x(x,t)}{\phi(x,t)} \\ &= \frac{\int_{-\infty}^{\infty} \frac{x-y}{t} \mathrm{e}^{-\frac{(x-y)^2}{4\nu t}}\phi_0(y)dy}{\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{(x-y)^2}{4\nu t}}\phi_0(y)dy}. \end{aligned} \tag{1.2.16}$$

Further, Hopf studied the behavior of the solution $u(x,t)$ of (1.2.10)-(1.2.11) in two ways:

1. The behavior of the solution as $t \to \infty$ while the viscosity ν stays constant.

2. The behavior of the solution as $\nu \to 0$ while keeping x and t are fixed.

After the discovery of Cole-Hopf transformation and this remarkable work by Hopf, many researchers started studying the higher order asymptotics for solutions of Burgers equation and its generalization. The motivation to study the large time asymptotic behavior of solutions to Burgers equation (1.2.10) is as follows:
Though the exact solution (1.2.15) of (1.2.12)-(1.2.13) is explicitly available, but for most of the initial data (1.2.13), the evaluation of the integral involved in (1.2.15) is tedious. Hence one has to look for either numerics or asymptotic analysis. Further, to obtain the solution of the Cauchy problem for Burgers equation (1.2.10)-(1.2.11), one can imagine the difficulty in evaluating the integrals involved in numerator and denominator of (1.2.16).

Zingano (2005) computed the limit

$$\lim_{t\to\infty} t^{\frac{1}{2}\left(1-\frac{1}{p}\right)} \left\|u(\cdot,t)\right\|_p, \quad 1 \leqslant p \leqslant \infty,$$

for solution $u(\cdot,t)$ of the generalized Burgers equation

$$u_t + au_x + buu_x = cu_{xx}$$

where $a,\ b,\ c > 0$ are real constants, subject to the initial data $u_0 \in L^1(\mathbb{R})$. Kloosterziel (1990) studied the large time asymptotic behavior of a Cauchy problem for the heat equation (1.2.12) by expanding its solution in terms of self-similar solutions for heat equation that form a basis for $L^2(\mathbb{R}, \mathrm{e}^{\frac{1}{2}x^2})$. He assumed the initial data to be a square integrable function with respect to the exponentially growing weight functions $\mathrm{e}^{\frac{1}{2}x^2}$, which allowed to represent the initial data as the linear combination of these similarity solutions. This approach revealed the large time behavior very quickly. In the large time asymptotic study for heat equation (1.2.12) done by Duoandikoetxea and Zuazua (1992), they considered an approximation to the solution ϕ of the Cauchy problem (1.2.12)-(1.2.13) which is of the form (the original multi-dimensional form is written in one-dimensional form)

$$\psi_{2k}(x,t) = \sum_{j=0}^{2k-1} \frac{(-1)^j}{j!} \mathcal{M}_j \partial_x^j G_t(x),$$

where $G_t(x) = \dfrac{1}{\sqrt{4\pi t}} \mathrm{e}^{-\frac{x^2}{4t}}$ is the heat kernel, $\mathcal{M}_j$ is the j-th moment of initial data ϕ_0 and ∂_x^j is the j-th partial derivative with respect to x. They showed that ψ_{2k} converges to the solution ϕ with a convergence order $O(t^{\frac{1}{2p}-\frac{2k+1}{2}})$ as $t \to \infty$ with respect to L^p-norm for $1 \leqslant p \leqslant \infty$. Witelski and Bernoff (1998) investigated the self-similar asymptotics for linear heat equation and its nonlinear generalization, the porous medium equation. By assuming the initial data to be non-negative, integrable, compactly supported, they introduced total mass, center of mass (space shift) and variance (time shift) in the self similar solutions to obtain large time asymptotics. Kim and Ni (2009) used the theory of truncated moment problem to prove the existence and the uniqueness of ρ_i's and c_i's that solve

$$\lim_{t\to 0} \int_{\mathbb{R}} x^j \phi_n(x,t) dx = \gamma_j, \quad j = 0, 1, \cdots, 2n-1,$$

where $\gamma_j = \int_{\mathbb{R}} x^j \phi_0(x) dx$ and

$$\phi_n(x,t) = \sum_{i=1}^{n} \frac{\rho_i}{\sqrt{4\pi t}} \mathrm{e}^{-\frac{(x-c_i)^2}{4t}},$$

is the approximation for the solution $\phi(x,t)$ of (1.2.12)-(1.2.13) with ϕ_0 satisfying $x^{2n}\phi_0(x) \in L^1(\mathbb{R})$ and $\phi_0 > 0$. In this way, they made the first $2n$ initial moments of ϕ_n to agree with those of ϕ. This agreement of the moments led them to obtain the same convergence order as obtained by Duoandikoetxea and Zuazua (1992). Kim (2011) developed a theory to generalize the truncated moment problem discussed in (Kim and Ni, 2009) to a complex measure space, which allowed to drop the positivity restriction of the initial data that was assumed in (Kim and Ni, 2009). He took

$$\varphi^n(x,t) = Re\left(\sum_{i=1}^{n} \frac{\rho_i}{\sqrt{4\pi(t+t_0)}} \mathrm{e}^{-\frac{(x-c_i)^2}{4(t+t_0)}} \right), \quad t > 0, \ x \in \mathbb{R},$$

to approximate the solution $\phi(x,t)$ of (1.2.12)-(1.2.13). As an application of generalized truncated moment problem he showed that $\phi(x,t)$ and $\varphi^n(x,t)$ share the same moments upto order $2n-1$ all the time by assuming $x^{2n}\phi_0(x) \in L^1(\mathbb{R})$ and ϕ_0 is bounded on $\mathbb{R}$.

Grundy (1983) studied the large time asymptotic behavior of solutions to a non-linear convective diffusion equation

$$u_t - (u^n)_x = (u^m)_{xx}, \quad x \in \mathbb{R}, \; t > 0, \tag{1.2.17}$$

subject to the initial data with compact support. It is to be noted that equation (1.2.17) is a generalization of the Burgers equation to non-linear diffusion. Kim and Tzavaras (2001) studied the effect of viscosity on the large time behavior of viscous Burgers equation. They derived the solution explicitly in terms of Hermite polynomials by using Cole-Hopf like transformation, which works for both zero mass as well as non zero mass initial profiles. Kim and Ni (2002) studied the large time asymptotic behavior of solution to (1.2.10)-(1.2.11) by assuming the initial data u_0 to be continuous with a compact support and sign changing. They used the Cole-Hopf transformation to reduce the viscous Burgers equation (1.2.10) to linear heat equation (1.2.12) and first they studied the asymptotics for heat equation by taking an approximation which is a heat kernel having a space shift and the strength given by total mass of the initial data. They proved that this approximation converges to the true solution with an order $O(t^{\frac{1}{2p}-\frac{3}{2}})$ as $t \to \infty$ in L^p-norm, for $1 \leqslant p \leqslant \infty$. These results obtained for heat equation are then converted to observe the asymptotics for Burgers equation. Miller and Bernoff (2003) constructed asymptotic approximate solution for the Cauchy problem for Burgers equation (1.2.10)-(1.2.11), with non-negative initial data u_0 which satisfies

$$\int_{\mathbb{R}} |x^3 u_0(x)| dx < \infty, \quad \text{and} \quad 0 < \int_{\mathbb{R}} u_0(x) dx < \infty.$$

The construction mainly depended on the agreement of first three initial moments of approximate solution to those of true solution of relevant heat equation (1.2.12). Further, they estimated that their approximate solution differs from the true solution by an error whose L^p-norm is of order $O(t^{\frac{1}{2p}-2})$ as $t \to \infty$, where $1 \leqslant p \leqslant \infty$. This work was an improvement over the work of Chern and Liu (1987) by a factor of $\frac{1}{t}$. Rao and Satyanarayana (2010) considered the viscous

Burgers equation (1.2.10) subject to the initial data, which is continuous, non-negative and square integrable function with respect to the exponentially growing weight functions $e^{\frac{1}{2}x^2}$ on $\mathbb{R}$. By following the approach of Kloosterziel (1990) they constructed single hump solutions in terms of the self similar solutions of heat equation. Later they compared their constructed solutions with the asymptotic solution of Miller and Bernoff (2003). Chung et al. (2010) introduced an asymptotic approximate solution for Burgers equation with a bounded initial value u_0 such that $x^{2n}u_0(x) \in L^1(\mathbb{R})$, by taking the inverse of Cole-Hopf transformation of a summation of n heat kernels. And they showed that k-th order moments of the exact and the approximate solution are contracting with an order $O((\sqrt{t})^{k-2n-1+\frac{1}{p}})$ as $t \to \infty$ in L^p-norm, for $1 \leqslant p \leqslant \infty$. To obtain higher order asymptotic behavior for Burgers equation, Yanagisawa (2007) proposed an asymptotic approximate solution to the relevant heat equation which is a combination of $k+1$ terms, each having derivative of heat kernel multiplied by the moment of the initial data. They introduced space shift and time shift in the leading order term of the approximate solution to precisely capture the effect of initial data on the long time behavior of the true solution. They proved that, as $t \to \infty$, the proposed asymptotic approximate solution converges to the true solution with an order $O(t^{\frac{1}{2p}-2-\frac{k}{2}})$ when time shift is positive and with order $O(t^{\frac{1}{2p}-\frac{3}{2}-\frac{k}{2}})$ when time shift is zero in L^p-norm, $1 \leqslant p \leqslant \infty$.

The general forced Burgers equation is given by

$$u_t + uu_x = \nu u_{xx} + f(x,t), \quad x \in \mathbb{R},\ t > 0, \tag{1.2.18}$$

where the function $f(x,t)$ is known as the forcing term. In recent decades, equation (1.2.18) has attracted much interest due to the work of Polyakov (1995). We refer to ((Xu et al., 2007) and the references there in) for various applications of (1.2.18). Ding et al. (2001) considered a forced Burgers equation, where the forcing term is only a function of x, given by

$$u_t + uu_x = \nu u_{xx} + kx, \quad x \in \mathbb{R}, t > 0, \tag{1.2.19}$$

with the initial data

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \tag{1.2.20}$$

where ν and k are positive constants. When $k = 0$, (1.2.19) is the well known Burgers equation (1.2.10). By assuming that $u_0 \in L^1_{loc}(\mathbb{R})$ and $\int^x u_0(r)dr = o(x^2)$ as $|x| \to \infty$, they used Cole-Hopf transformation to reduce (1.2.19) to a linear partial differential equation

$$\phi_t = \nu\phi_{xx} - \frac{x^2}{\nu}\phi, \quad x \in \mathbb{R}, \quad t > 0. \tag{1.2.21}$$

By obtaining the solution of (1.2.21) in terms of Hermite polynomials, they studied the large time asymptotic behavior for (1.2.19) under the Cole-Hopf transformation. Bec and Khanin (2003) studied a forced Burgers equation in an unbounded domain. Eule and Friedrich (2006) solved a Cauchy problem for

$$u_t + uu_x = \nu u_{xx} + G(t)x, \quad x \in \mathbb{R},\ t > 0, \tag{1.2.22}$$

by assuming $G(t)$ to be a white noise force. Here, it is to be observed that the forcing term is linear in spatial coordinates. Xu et al. (2007) considered a generalization of (1.2.18), namely,

$$u_t + a(x,t)uu_x + b(x,t)u_{xx} = f(x,t), \tag{1.2.23}$$

where $a(x,t), b(x,t)$ and $f(x,t)$ are all real functions of x and t. To linearize (1.2.23) to the heat equation, they proposed a generalized Cole-Hopf transformation of the form

$$u(x,t) = \alpha(x,t)\frac{\phi_\xi(\xi,\tau)}{\phi(\xi,\tau)} + \beta(x,t), \quad \xi = \xi(x,t), \quad \tau = \tau(t), \tag{1.2.24}$$

where $\alpha(x,t) \neq 0, \xi \neq 0$, and the determination of $\alpha(x,t), \beta(x,t), \xi(x,t)$ and $\tau(t)$ is based on certain constraints on $a(x,t), b(x,t)$ and $f(x,t)$. Salas (2010) transformed a generalized Burgers equation of the form

$$u_t + \alpha uu_x + \beta u_{xx} = G(t)x, \tag{1.2.25}$$

to the standard heat equation by constructing a generalized Cole-Hopf transformation. They gave the exact solutions for (1.2.25) by travelling wave method. Rao and Yadav (2010a) studied the large time behavior for the forced Burgers equation (1.2.19) subject to the initial data u_0 that is bounded and compactly supported. For which, they used Cole-Hopf transformation to linearize (1.2.19), and then a transformation to reduce that linearized equation to heat equation. Following the approach of Kloosterziel (1990), they constructed self similar solution to (1.2.19) which revealed the large time behavior very quickly. Rao and Yadav (2010c) studied the forced Burgers equation which is a generalization of (1.2.19), given by

$$u_t + uu_x = u_{xx} + \frac{kx}{(2\beta t+1)^2}, \quad x \in \mathbb{R}, \quad t > 0, \tag{1.2.26}$$

subject to the initial data u_0 satisfying

$$\exp\left(-\frac{1}{2}\int^x u_0(y)dy\right) \in L^2(\mathbb{R}, \mathrm{e}^{\frac{\beta x^2}{2}}).$$

By making use of Cole-Hopf transformation, they reduced (1.2.26) to a linear partial differential equation

$$\phi_t = \phi_{xx} - \frac{kx^2}{4(2\beta t+1)^2}. \tag{1.2.27}$$

They constructed a family of self-similar solutions to (1.2.27) which forms an orthonormal basis for $L^2(\mathbb{R}, \mathrm{e}^{\frac{\beta x^2}{2}})$ at $t = 0$. They represented solution ϕ in terms of family of self-similar solutions. Under the Cole-Hopf transformation they obtained solution u from ϕ and investigated the large time behavior. Since the work of Rao and Yadav (2010c) supports only unbounded initial profiles, Rao and Yadav (2010b) considered the same equation (1.2.26) with bounded and compactly supported initial profiles. Yadav (2013) obtained explicit solutions of a system of forced Burgers equations subject to some classes of bounded and compactly supported initial data and also subject to certain unbounded initial data.

1.3 Preliminaries

We use the following notations for spaces in the thesis.

1. $C(\mathbb{R})$ denotes the space of all continuous functions $f : \mathbb{R} \to \mathbb{R}$.

2. $C^\infty(\mathbb{R})$ denotes the space of infinitely differentiable functions $f : \mathbb{R} \to \mathbb{R}$.

3. $L^p(\mathbb{R})$ for $1 \leqslant p < \infty$, denotes the space of all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$\|f\|_{L^p(\mathbb{R})} := \left(\int_{\mathbb{R}} |f(x)|^p dx\right)^{\frac{1}{p}} < \infty.$$

4. $L^\infty(\mathbb{R})$ denotes the space of all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$\|f\|_{L^\infty(\mathbb{R})} := \operatorname*{ess\,sup}_{x \in \mathbb{R}} |f(x)| < \infty.$$

5. $L^2(\mathbb{R}, \mathrm{e}^{\frac{1}{2}x^2})$ denotes the space of all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$\left(\int_{\mathbb{R}} \mathrm{e}^{\frac{1}{2}x^2} (f(x))^2 dx\right)^{\frac{1}{2}} < \infty.$$

6. Let $\Omega \subset \mathbb{R}$ is open, $L^1_{loc}(\Omega)$ denotes the space of all functions $f : \Omega \to \mathbb{R}$ that satisfy

$$\int_K f(x) dx < \infty \quad \text{for every compact set } K \subseteq \Omega.$$

The following definitions, lemma and theorem can be found in (Titchmarsh, 1986), (Kesavan, 1989), (Evans, 1998) and (Stavroulakis and Tersian, 2004).

Theorem 1.3.1. *Assume $\phi_0 \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\phi(x,t)$ is defined by*

$$\phi(x,t) = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\nu t}} \phi_0(y) dy.$$

Then

1. $\phi \in C^\infty(\mathbb{R} \times (0, \infty))$,
2. $\phi_t = \nu \phi_{xx}, \quad x \in \mathbb{R}, \quad t > 0,$ *and*
3. $\lim_{(x,t)\to(x_0,0)} \phi(x,t) = \phi_0(x_0)$ *for each* $x_0 \in \mathbb{R}$.

Definition 1.3.2. *Big-oh notation:* *We write $f = O(g)$ as $x \to x_0$, provided that there exists a constant C such that*

$$|f(x)| \leqslant C|g(x)|,$$

for all x sufficiently close to x_0.

Definition 1.3.3. ***Little-oh notation:*** *We write* $f = o(g)$ as $x \to x_0$, *provided*

$$\lim_{x\to x_0} \frac{|f(x)|}{|g(x)|} = 0.$$

Definition 1.3.4. *The* ***j-th moment*** *of any function* f, *where* $j \geqslant 0$ *is an integer, is defined as*

$$\int_{\mathbb{R}} x^j f(x)dx.$$

Lemma 1.3.5. *Suppose that* $(1+|x|)^k f \in L^1(\mathbb{R})$. *Then the equality*

$$\begin{aligned}&\int_{-\infty}^{x_0}\left(\int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_{k-1}}\left(\int_{-\infty}^{x_k} f(x_{k+1})dx_{k+1}\right)dx_k\cdots\right)dx_1\\ =& \sum_{j=0}^{k}\frac{(-1)^{k-j}}{k!}\overline{\mathcal{M}}_{k-j}(x_0)\frac{k!\ x_0^j}{j!\ (k-j)!},\end{aligned}$$

holds with $\overline{\mathcal{M}}_j(x) = \int_{-\infty}^{x} y^j f(y)dy$. *Here* j *and* k *are non negative integers.*

Definition 1.3.6. *Rodrigues formula for* n^{th}*-degree* ***Hermite polynomial*** *is defined as*

$$H_n(x) = (-1)^n \mathrm{e}^{x^2}\frac{d^n}{dx^n}\mathrm{e}^{-x^2}.$$

Definition 1.3.7. ***Holder's inequality:*** *Assume* $1 \leqslant p\ , q \leqslant \infty$, $\frac{1}{p}+\frac{1}{q} = 1$. *Then if* $u \in L^p(\Omega)$, $v \in L^q(\Omega)$, *we have*

$$\int_{\Omega}|uv|dx \leqslant \|u\|_{L^p(\Omega)}\|v\|_{L^q(\Omega)}.$$

Definition 1.3.8. ***Young's inequality****: Let* $1 \leqslant p,\ q,\ r < \infty$ *such that* $\frac{1}{p}+\frac{1}{q} = \frac{1}{r}+1$. *If* $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$ *then* $f*g \in L^r(\mathbb{R})$ *and*

$$\|f*g\|_{L^r(\mathbb{R})} \leqslant \|f\|_{L^p(\mathbb{R})}\|g\|_{L^q(\mathbb{R})}.$$

Here $*$ *represents the convolution.*

1.4 Organization of the thesis

This thesis is organized as follows:

In Chapter 2, we study the large time asymptotics of solutions to the heat equation subject to the initial data ϕ_0 that is assumed to satisfy $(1+|x|)^{k+3+\epsilon}\phi_0 \in$

$L^1(\mathbb{R})$, for any positive integer k and a small real number $\epsilon > 0$. An approximate solution for the Cauchy problem is proposed as the combination of derivatives of heat kernel. In this approximation, we incorporate the moments of initial data and introduce spatial shifts into the leading order term as well as penultimate term. The construction of the approximate solution mainly depends on the agreement of the moments of it with those of the true solution as $t \to 0^+$. As an outcome of this, we show that the constructed approximation has higher order of convergence. We extend these results to observe asymptotic behavior of solutions to the viscous Burgers equation. Finally, we compare the constructed approximation to the earlier proposed approximation by Yanagisawa (2007).

In Chapter 3, we deal with the Cauchy problem for a forced Burgers equation (FBE) with an assumption that, the initial data u_0 is continuous on $\mathbb{R}$ and $u_0 \in L^1(\mathbb{R})$. We study the large time asymptotic behavior of solutions to the considered Cauchy problem. For which, we introduce a Cole-Hopf like transformation to linearize the FBE and use the method of separation of variables to construct solutions of FBE with a different class of initial data. Later, we prove the existence of solution to the considered Cauchy problem. We also give an asymptotic approximation to Cauchy problem for FBE and show that, as $t \to \infty$, the magnitude of the difference function of the true and approximate solutions decays algebraically. Chapter 4 sets forth the conclusion of thesis.

Chapter 2

Asymptotic Behavior of Solutions to the Diffusion Equation

2.1 Introduction

In this chapter, we study the large time asymptotics to the solutions of the Cauchy problem

$$\phi_t = \phi_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \tag{2.1.1}$$

$$\phi(x,0) = \phi_0(x), \quad x \in \mathbb{R}, \tag{2.1.2}$$

with the assumption that the initial data ϕ_0 satisfies

$$(1+|x|)^{k+3+\epsilon}\phi_0 \in L^1(\mathbb{R}), \tag{2.1.3}$$

for any positive integer k and a small real number $\epsilon > 0$. This work is continuation to the work done by Yanagisawa (2007).

We denote the *j-th order moment* of function ϕ_0, where j is a non negative integer, by

$$\mathcal{M}_j := \int_{\mathbb{R}} x^j \phi_0(x)dx.$$

From (2.1.3), it is clear that $\mathcal{M}_j < \infty$ for $0 \leqslant j \leqslant k+2$.

It is well known (Hopf, 1950) that Cole-Hopf transformation reduces the viscous Burgers equation to heat equation. Hence, for studying the asymptotic behavior of solutions to the Burgers equation, one may first investigate the asymptotic behavior of solutions to the heat equation. Chern and Liu (1987) studied large time asymptotics of solutions to heat equation by constructing a self similar solution, which converges to the true solution with order $O(t^{-1+\frac{1}{2p}})$ in L^p norm when $t \to \infty$. If $\phi_0(x)\mathrm{e}^{|x|}$ is bounded on $\mathbb{R}$, the decay rate of the solution $u(x,t)$ of (2.1.1)-(2.1.2) is given in (Rubinstein and Rubinstein, 1998) by

$$\begin{aligned}\phi(x,t) \asymp\ & \frac{1}{2\sqrt{\pi t}}\mathrm{e}^{-\frac{x^2}{4t}}\left[M_0 + t^{-1}\left(\frac{xM_1}{2} - \frac{M_2}{4}\right)\right.\\ & \left. + \frac{1}{32}t^{-2}\left(4x^2M_2 - 4xM_3 + M_4\right) + O(t^{-3})\right], \qquad \text{as} \ \ t \to \infty,\end{aligned}$$

when x is bounded on $\mathbb{R}$. Witelski and Bernoff (1998) considered the heat equation with nonnegative compactly supported integrable initial functions. They introduced three parameters into the approximate solution, these correspond to the mass, center of mass and variance of the given initial data. Miller and Bernoff (2003) studied the heat equation with nonnegative piecewise initial data assuming that third order moment of initial data exists. Using the approximation obtained in (Witelski and Bernoff, 1998), Miller and Bernoff (2003) derived the rate of convergence to be of order $O(t^{-2+\frac{1}{2p}})$ in L^p norm when $t \to \infty$. Kloosterziel (1990) considered the initial data, which is nonnegative and square integrable with respect to weight function $\mathrm{e}^{\frac{x^2}{2}}$ on $\mathbb{R}$. They then expanded the solution of the heat equation in terms of its self-similar solutions. This expansion reveals the large time asymptotics quickly. Rao and Satyanarayana (2010) studied the relations between the approximations given by Kloosterziel (1990) and Miller and Bernoff (2003). Assuming $x^{2n}\phi_0 \in L^1(\mathbb{R})$, ϕ_0 is bounded and not changing its sign on $\mathbb{R}$, Kim and Ni (2009) constructed an asymptotic solution

$$\phi_n(x,t) = \sum_{i=1}^{n} \frac{\rho_i}{\sqrt{4\pi t}}\mathrm{e}^{-\frac{-(x-c_i)^2}{4t}},$$

to (2.1.1)-(2.1.2), where ρ_i and c_i are constants to be determined based on the moments of ϕ_0, showing that

$$||\phi(.,t) - \phi_n(.,t)||_p = O(t^{-\frac{2n+1}{2}+\frac{1}{2p}}), \quad t \to \infty.$$

Kim (2011) generalized the theory of the truncated moment problem to include complex measures and then obtained the large time behavior of solutions to heat equation by constructing the approximations as linear combination of heat kernels. Using the generalization of truncated moment problem (Kim, 2011), Satynarayana et al. (2017) studied the N-wave solutions for heat equation. Duoandikoetxea and Zuazua (1992) constructed an approximate solution for heat equation in the following form;

$$\psi_{2n}(x,t) = \sum_{j=0}^{2n-1} (-1)^j \frac{\gamma_i}{\sqrt{4\pi t}} \partial_x^j \left(e^{-\frac{x^2}{4t}} \right) \quad \text{for} \quad t > 0, \ x \in \mathbb{R}, \tag{2.1.4}$$

They showed that the approximate solution ψ_{2n} converges to the solution ϕ with an order $O(t^{-\frac{2n+1}{2}+\frac{1}{2p}})$ in L^p-norm when $t \to \infty$.

Generalizing the works of Duoandikoetxea and Zuazua (1992) and Miller and Bernoff (2003), Yanagisawa (2007) introduced an asymptotic approximate solution

$$\phi^k(x,t) = \sum_{j=0}^{k-1} (-1)^j \frac{\mathcal{M}_j}{j!} \partial_x^j G_t(x) + (-1)^k \frac{\mathcal{M}_k}{k!} \partial_x^k G_{t+(t_k)_+}(x - \gamma_k), \quad \text{for} \quad t > 0, \ x \in \mathbb{R}, \tag{2.1.5}$$

to the Cauchy problem (2.1.1)-(2.1.2), where G_t is the heat kernel given by

$$G_t(z) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{z^2}{4t} \right).$$

They chose space shift γ_k and time shift $(t_k)_+ = \max(t_k, 0)$ by making $(k+1)^{th}$ and $(k+2)^{th}$ moments of $\phi(x,t) - \phi^k(x,t)$ to vanish when $t \to 0^+$. As $t \to \infty$, they derived the convergence order to be $O(t^{(\frac{1}{2p}-2-\frac{k}{2})})$ when $(t_k)_+ > 0$ and $O(t^{(\frac{1}{2p}-\frac{3}{2}-\frac{k}{2})})$ when $(t_k)_+ = 0$ with respect to L^p-norm, $1 \leqslant p \leqslant \infty$.

Our main results are stated herewith:

Theorem 2.1.1. *Let $\phi(x,t)$ be a solution to heat equation (2.1.1) subject to (2.1.2) satisfying $(1+|x|)^{k+3+\epsilon}\phi_0 \in L^1(\mathbb{R})$, where k is any positive integer and ϵ is any small positive real number. Then, for $1 \leqslant p \leqslant \infty$, we have*

$$||\phi(.,t) - \phi^k(.,t)||_{L^p(\mathbb{R})} = O(t^{-\frac{4+k}{2}+\frac{1}{2p}}), \quad t \to \infty, \tag{2.1.6}$$

where ϕ^k *is given by*

$$\phi^k(x,t) = \begin{cases} \sum_{j=0}^{k-2}(-1)^j \frac{\mathcal{M}_j}{j!}\partial_x^j G_t(x) + (-1)^{(k-1)}\frac{\mathcal{M}_{k-1}}{(k-1)!}\partial_x^{k-1}G_t\left(x - \frac{\mathcal{M}_k}{k\mathcal{M}_{k-1}}\right) \\ \qquad \textit{if } \mathcal{M}_{k-1} \neq 0, \mathcal{M}_{k-1}\mathcal{M}_{k+1} = \frac{k+1}{2k}\mathcal{M}_k^2, \frac{6k^2}{(k+2)(k+1)}\mathcal{M}_{k-1}^2\mathcal{M}_{k+2} = \mathcal{M}_k^3, \\ \sum_{j=0}^{k-2}(-1)^j \frac{\mathcal{M}_j}{j!}\partial_x^j G_t(x) + (-1)^{(k-1)}\frac{\mathcal{M}_{k-1}}{(k-1)!}\partial_x^{k-1}G_t(x - b_0) \\ + \frac{(-1)^k}{k!}[\mathcal{M}_k - k\mathcal{M}_{k-1}b_0]\partial_x^k G_t\left(x - \frac{2\mathcal{M}_{k+1} - k(k+1)\mathcal{M}_{k-1}b_0^2}{2(k+1)(\mathcal{M}_k - k\mathcal{M}_{k-1}b_0)}\right) \\ \qquad \textit{if } \ \mathcal{M}_{k-1} \neq 0, \\ \sum_{j=0}^{k-2}(-1)^j \frac{\mathcal{M}_j}{j!}\partial_x^j G_t(x) + (-1)^k\frac{\mathcal{M}_k}{k!}\partial_x^k G_t\left(x - \frac{\mathcal{M}_{k+1}}{(k+1)\mathcal{M}_k}\right) \\ \qquad \textit{if } \mathcal{M}_{k-1} = 0, \mathcal{M}_k \neq 0, \mathcal{M}_{k+2}\mathcal{M}_k = \frac{k+2}{2(k+1)}[\mathcal{M}_{k+1}]^2, \end{cases} \tag{2.1.7}$$

where $b_0(\neq \frac{\mathcal{M}_k}{k\mathcal{M}_{k-1}})$ *is any solution of the equation*

$$\begin{aligned} &[k^2(k+1)(k+2)\mathcal{M}_{k-1}^2]b_0^4 - [4k(k+1)(k+2)\mathcal{M}_{k-1}\mathcal{M}_k]b_0^3 \\ &+[12k(k+2)\mathcal{M}_{k-1}\mathcal{M}_{k+1}]b_0^2 - [24k\mathcal{M}_{k-1}\mathcal{M}_{k+2}]b_0 \\ &+12[2\mathcal{M}_k\mathcal{M}_{k+2} - \frac{(k+2)}{(k+1)}\mathcal{M}_{k+1}^2] = 0. \end{aligned} \tag{2.1.8}$$

The existence of real valued solution of the quartic equation (2.1.8) is discussed in Remark 2.2.7.

This chapter is organized as follows. In section 2.2, we construct an approximate solution to the Cauchy problem (2.1.1)-(2.1.2) and then extend these results to viscous Burgers equation. Section 2.3 presents the advantages and disadvantages of the proposed asymptotic approximate solution by comparing with Yanagisawa (2007)'s asymptotic approximate solution (2.1.5).

2.2 An asymptotic approximation

In this section, we propose an asymptotic approximate solution $\phi^k(x,t)$ to the solution of the initial value problem (2.1.1)-(2.1.2) by

$$\phi^k(x,t) = \sum_{j=0}^{k-2}(-1)^j \frac{\mathcal{M}_j}{j!}\partial_x^j G_t(x) + a_0\partial_x^{k-1}G_t(x - b_0) + a_1\partial_x^k G_t(x - b_1), \tag{2.2.9}$$

where the arbitrary constants a_0, a_1, b_0 and b_1 are to be chosen appropriately. Here $G_t(z)$ represents one-dimensional heat kernel

$$G_t(z) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{z^2}{4t}\right).$$

As an outcome of this, we will obtain higher order convergence of ϕ^k as mentioned in the Theorem 2.1.1.

We now prove few Lemmas which are needed to prove Theorem 2.1.1.

Lemma 2.2.1. *Let ϕ be the solution to the problem (2.1.1)-(2.1.2) and ϕ^k be an asymptotic approximation given in (2.2.9), where k is any positive integer. Then we have the following:*

$$\lim_{t\to 0^+} \int_{\mathbb{R}} x^{k-1}[\phi(x,t) - \phi^k(x,t)]dx = \mathcal{M}_{k-1} - (-1)^{k-1}(k-1)!a_0, \tag{2.2.10}$$

$$\lim_{t\to 0^+} \int_{\mathbb{R}} x^{k}[\phi(x,t) - \phi^k(x,t)]dx = \mathcal{M}_{k} - (-1)^{k-1}k!a_0b_0 - (-1)^k k!a_1, \tag{2.2.11}$$

$$\lim_{t\to 0^+} \int_{\mathbb{R}} x^{k+1}[\phi(x,t) - \phi^k(x,t)]dx = \mathcal{M}_{k+1} - (-1)^{k-1}\frac{(k+1)!}{2}a_0b_0^2 - (-1)^k(k+1)!a_1b_1, \tag{2.2.12}$$

$$\lim_{t\to 0^+} \int_{\mathbb{R}} x^{k+2}[\phi(x,t) - \phi^k(x,t)]dx = \mathcal{M}_{k+2} - (-1)^{k-1}\frac{(k+2)!}{3!}a_0b_0^3 - (-1)^k\frac{(k+2)!}{2}a_1b_1^2. \tag{2.2.13}$$

Proof. Using integration by parts, we have

$$\lim_{t\to 0^+} \int_{\mathbb{R}} x^j \partial_x^l G_t(x)dx = (-1)^l j!\delta_{lj}, \quad \text{for any non-negative integers } l \text{ and } j, \tag{2.2.14}$$

where δ_{lj} represents Kronecker's delta and $G_t(s) = \frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{s^2}{4t}\right)$. We first prove (2.2.10). Making use of (2.2.14), it is easy to see that

$$\begin{aligned}\lim_{t\to 0^+} \int_{\mathbb{R}} x^{k-1}\partial_x^{k-1} G_t(x-b_0)dx &= \lim_{t\to 0^+} \int_{\mathbb{R}} (y+b_0)^{k-1}\partial_y^{k-1} G_t(y)dy \\ &= (-1)^{k-1}(k-1)!, \end{aligned} \tag{2.2.15}$$

and

$$\begin{aligned}\lim_{t\to 0^+} \int_{\mathbb{R}} x^{k-1}\partial_x^{k} G_t(x-b_1)dx &= \lim_{t\to 0^+} \int_{\mathbb{R}} (y+b_1)^{k-1}\partial_y^{k} G_t(y)dy \\ &= 0. \end{aligned} \tag{2.2.16}$$

It then follows from (2.2.14)-(2.2.16) that

$$\begin{aligned}
&\lim_{t\to 0^+}\int_{\mathbb{R}} x^{k-1}[\phi(x,t)-\phi^k(x,t)]dx \\
&= \mathcal{M}_{k-1} - a_0 \lim_{t\to 0^+}\int_{\mathbb{R}} x^{k-1}\partial_x^{k-1}G_t(x-b_0)dx - a_1 \lim_{t\to 0^+}\int_{\mathbb{R}} x^{k-1}\partial_x^{k}G_t(x-b_1)dx \\
&= \mathcal{M}_{k-1} - (-1)^{k-1}(k-1)!a_0,
\end{aligned}$$

which proves (2.2.10).

We now proceed to prove (2.2.11). In view of (2.2.14), we have

$$\begin{aligned}
\lim_{t\to 0^+}\int_{\mathbb{R}} x^{k}\partial_x^{k-1}G_t(x-b_0)dx &= \lim_{t\to 0^+}\int_{\mathbb{R}} (y+b_0)^{k}\partial_y^{k-1}G_t(y)dy \\
&= (-1)^{k-1}(k-1)!\binom{k}{1}b_0 \\
&= (-1)^{k-1}k!b_0, \qquad (2.2.17)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{t\to 0^+}\int_{\mathbb{R}} x^{k}\partial_x^{k}G_t(x-b_1)dx &= \lim_{t\to 0^+}\int_{\mathbb{R}} (y+b_1)^{k}\partial_y^{k}G_t(y)dy \\
&= (-1)^{k}k!. \qquad (2.2.18)
\end{aligned}$$

Now (2.2.11) can be derived from (2.2.14),(2.2.17) and (2.2.18) as follows:

$$\begin{aligned}
&\lim_{t\to 0^+}\int_{\mathbb{R}} x^{k}[\phi(x,t)-\phi^k(x,t)]dx \\
&= \mathcal{M}_{k} - a_0 \lim_{t\to 0^+}\int_{\mathbb{R}} x^{k}\partial_x^{k-1}G_t(x-b_0)dx - a_1 \lim_{t\to 0^+}\int_{\mathbb{R}} x^{k}\partial_x^{k}G_t(x-b_1)dx \\
&= \mathcal{M}_{k} - (-1)^{k-1}k!a_0b_0 - (-1)^{k}k!a_1.
\end{aligned}$$

Next, using (2.2.14), we see that

$$\begin{aligned}
\lim_{t\to 0^+}\int_{\mathbb{R}} x^{k+1}\partial_x^{k-1}G_t(x-b_0)dx &= \lim_{t\to 0^+}\int_{\mathbb{R}} (y+b_0)^{k+1}\partial_y^{k-1}G_t(y)dy \\
&= (-1)^{k-1}(k-1)!\binom{k+1}{2}b_0^2 \\
&= (-1)^{k-1}\frac{(k+1)!}{2}b_0^2, \qquad (2.2.19)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{t\to 0^+}\int_{\mathbb{R}} x^{k+1}\partial_x^{k}G_t(x-b_1)dx &= \lim_{t\to 0^+}\int_{\mathbb{R}} (y+b_1)^{k+1}\partial_y^{k}G_t(y)dy \\
&= (-1)^{k}k!\binom{k+1}{1}b_1 \\
&= (-1)^{k}(k+1)!b_1. \qquad (2.2.20)
\end{aligned}$$

Therefore, from (2.2.14), (2.2.19)-(2.2.20), we get

$$\begin{aligned}
&\lim_{t\to 0^+}\int_{\mathbb{R}} x^{k+1}[\phi(x,t)-\phi^k(x,t)]dx \\
&= \mathcal{M}_{k+1} - a_0 \lim_{t\to 0^+}\int_{\mathbb{R}} x^{k+1}\partial_x^{k-1}G_t(x-b_0)dx - a_1 \lim_{t\to 0^+}\int_{\mathbb{R}} x^{k+1}\partial_x^{k}G_t(x-b_1)dx \\
&= \mathcal{M}_{k+1} - (-1)^{k-1}\frac{(k+1)!}{2}a_0 b_0^2 - (-1)^k (k+1)! a_1 b_1,
\end{aligned}$$

which verifies the equality (2.2.12).

Finally to prove (2.2.13), we use (2.2.14) to find that

$$\begin{aligned}
\lim_{t\to 0^+}\int_{\mathbb{R}} x^{k+2}\partial_x^{k-1}G_t(x-b_0)dx &= \lim_{t\to 0^+}\int_{\mathbb{R}} (y+b_0)^{k+2}\partial_y^{k-1}G_t(y)dy \\
&= (-1)^{k-1}(k-1)!\binom{k+2}{3} b_0^3 \\
&= (-1)^{k-1}\frac{(k+2)!}{3!} b_0^3, \qquad (2.2.21)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{t\to 0^+}\int_{\mathbb{R}} x^{k+2}\partial_x^{k}G_t(x-b_1)dx &= \lim_{t\to 0^+}\int_{\mathbb{R}} (y+b_1)^{k+2}\partial_y^{k}G_t(y)dy \\
&= (-1)^{k}k!\binom{k+2}{2} b_1^2 \\
&= (-1)^{k}\frac{(k+2)!}{2} b_1^2. \qquad (2.2.22)
\end{aligned}$$

Using (2.2.14), (2.2.21)-(2.2.22), it can be seen that

$$\begin{aligned}
&\lim_{t\to 0^+}\int_{\mathbb{R}} x^{k+2}[\phi(x,t)-\phi^k(x,t)]dx \\
&= \mathcal{M}_{k+2} - a_0 \lim_{t\to 0^+}\int_{\mathbb{R}} x^{k+2}\partial_x^{k-1}G_t(x-b_0)dx - a_1 \lim_{t\to 0^+}\int_{\mathbb{R}} x^{k+2}\partial_x^{k}G_t(x-b_1)dx \\
&= \mathcal{M}_{k+2} - (-1)^{k-1}\frac{(k+2)!}{3!}a_0 b_0^3 - (-1)^k \frac{(k+2)!}{2} a_1 b_1^2.
\end{aligned}$$

This completes the proof. □

Remark 2.2.2. *It is easy to see that the moments of initial function, upto the order $k-2$, were incorporated in the asymptotic approximate solution (2.2.9) as if*

$$\lim_{t\to 0^+}\int_{\mathbb{R}} x^{j}[\phi(x,t)-\phi^k(x,t)]dx = 0, \quad j = 0,1,2\ldots,k-2,$$

provided $k > 2$.

By virtue of above Lemma 2.2.1, we list out some key observations:

1. The choice of
$$a_0 := (-1)^{k-1}\frac{\mathcal{M}_{k-1}}{(k-1)!}, \tag{2.2.23}$$
coefficient in penultimate term of the asymptotic approximate solution (2.2.9), makes the $(k-1)^{th}$ order moment of ϕ to agree with that of ϕ^k as $t \to 0^+$.

2. If we choose
$$a_1 := \frac{(-1)^k}{k!}[\mathcal{M}_k - k\mathcal{M}_{k-1}b_0], \tag{2.2.24}$$
in (2.2.9) and a_0 as in (2.2.23), then k^{th} order initial moment of ϕ agrees with that of ϕ^k for any arbitrary constant b_0.

3. Choosing
$$b_1 := \frac{2\mathcal{M}_{k+1} - k(k+1)\mathcal{M}_{k-1}b_0^2}{2(k+1)(\mathcal{M}_k - k\mathcal{M}_{k-1}b_0)}, \tag{2.2.25}$$
in (2.2.9) and $a_0,\ a_1$ as in (2.2.23)-(2.2.24), we notice that $(k+1)^{th}$ order initial moment of ϕ agrees with that of ϕ^k for any arbitrary value of b_0.

4. Finally, if we assume that $(k+2)^{th}$ order initial moment of ϕ agree with that of ϕ^k, then from (2.2.13) we have
$$\mathcal{M}_{k+2} - (-1)^{k-1}\frac{(k+2)!}{3!}a_0b_0^3 - (-1)^k\frac{(k+2)!}{2}a_1b_1^2 = 0. \tag{2.2.26}$$
Substituting the expressions (2.2.23)-(2.2.25) of $a_0,\ a_1$ and b_1 in the equation (2.2.26), we get an equation in b_0;
$$\begin{aligned}&[k^2(k+1)(k+2)\mathcal{M}_{k-1}^2]b_0^4 - [4k(k+1)(k+2)\mathcal{M}_{k-1}\mathcal{M}_k]b_0^3\\ &+[12k(k+2)\mathcal{M}_{k-1}\mathcal{M}_{k+1}]b_0^2 - [24k\mathcal{M}_{k-1}\mathcal{M}_{k+2}]b_0\\ &+12[2\mathcal{M}_k\mathcal{M}_{k+2} - \frac{(k+2)}{(k+1)}\mathcal{M}_{k+1}^2] = 0.\end{aligned}$$
Hence, by assigning any solution of the above equation to b_0, we notice that $(k+2)^{th}$ order initial moment of ϕ also agrees with that of ϕ^k.

5. The conditions $\mathcal{M}_{k-1} = 0$ and
$$\lim_{t\to 0^+}\int_{\mathbb{R}} x^j[\phi(x,t) - \phi^k(x,t)]dx = 0, \qquad 0 \leqslant j < k+2,$$
reduce the proposed approximation (2.2.9) to (2.2.53).

Motivated by the work of Yanagisawa (2007), we now introduce a function $F(\phi_0)$ by

$$
\begin{aligned}
&F(\phi_0)(x) = \\
&\int_{-\infty}^{x}\left(\int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_{k+1}}\left(\int_{-\infty}^{x_{k+2}}\phi_0(x_{k+3})dx_{k+3}\right)dx_{k+2}\cdots\right)dx_1 \\
&-\sum_{j=0}^{k-2}(-1)^j\frac{\mathcal{M}_j}{j!}\int_0^x\left(\int_0^{x_1}\cdots\int_0^{x_{k-j}}\left(\int_0^{x_{k+1-j}}H_0(x_{k+2-j})dx_{k+2-j}\right)dx_{k+1-j}\cdots\right)dx_1 \\
&-a_0\int_{b_0}^{x}\left(\int_{b_0}^{x_1}\left(\int_{-\infty}^{x_2}H_{b_0}(x_3)dx_3\right)dx_2\right)dx_1 - a_1\int_{b_1}^{x}\left(\int_{-\infty}^{x_1}H_{b_1}(x_2)dx_2\right)dx_1,
\end{aligned}
$$

where $H_a(x)$ is the Heaviside's unit step function

$$
H_a(x) = \begin{cases} 0 & \text{if} \quad x \leqslant a, \\ 1 & \text{if} \quad x > a. \end{cases}
$$

The following lemma talks about the integrability of the function $F(\phi_0)$ on $\mathbb{R}$.

Lemma 2.2.3. *Assume that $\phi(x,t)$ is a solution to heat equation (2.1.1) subject to (2.1.2) satisfying $(1+|x|)^{k+3+\epsilon}\phi_0 \in L^1(\mathbb{R})$. Further, let*

$$
\lim_{t\to 0^+}\int_{\mathbb{R}} x^j[\phi(x,t)-\phi^k(x,t)]dx = 0, \qquad 0 \leqslant j \leqslant k+2, \tag{2.2.27}
$$

where $\phi^k(x,t)$ is given in (2.2.9).Then $F(\phi_0) \in L^1(\mathbb{R})$.

Proof. As $(1+|x|)^{k+3+\epsilon}\phi_0 \in L^1(\mathbb{R})$, it is obvious that $(1+|x|)^{k+2+\epsilon}\phi_0 \in L^1(\mathbb{R})$. Hence, using Lemma 1.3.5, we have

$$
\begin{aligned}
B_1(x) &:= \int_{-\infty}^{x}\left(\int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_{k+1}}\left(\int_{-\infty}^{x_{k+2}}\phi_0(x_{k+3})dx_{k+3}\right)dx_{k+2}\cdots\right)dx_1 \\
&= \sum_{j=0}^{k+2}\frac{(-1)^j}{j!}\overline{\mathcal{M}}_j(x)\frac{x^{k+2-j}}{(k+2-j)!},
\end{aligned} \tag{2.2.28}
$$

where $\overline{\mathcal{M}}_j(x) = \int_{-\infty}^{x} y^j\phi_0(y)dy$.

$$
\begin{aligned}
B_2(x) &:= -\sum_{j=0}^{k-2}(-1)^j\frac{\mathcal{M}_j}{j!} \\
&\quad\times\int_0^x\left(\int_0^{x_1}\cdots\int_0^{x_{k-j}}\left(\int_0^{x_{k+1-j}}H_0(x_{k+2-j})dx_{k+2-j}\right)dx_{k+1-j}\cdots\right)dx_1 \\
&= -\sum_{j=0}^{k-2}\frac{(-1)^j}{j!}\mathcal{M}_j\frac{x_+^{k+2-j}}{(k+2-j)!},
\end{aligned} \tag{2.2.29}
$$

where $x_+ = \max\{x, 0\}$. Consider

$$
\begin{aligned}
B_3(x) &:= -a_0 \int_{b_0}^{x} \left(\int_{b_0}^{x_1} \left(\int_{-\infty}^{x_2} H_{b_0}(x_3) dx_3 \right) dx_2 \right) dx_1 \\
&= \begin{cases} -a_0 \left[\frac{x^3}{3!} - \frac{b_0}{2} x^2 + \frac{b_0^2}{2} x - \frac{b_0^3}{3!} \right], & \text{if } \ x > b_0 \\ 0, & \text{if } \ x \leqslant b_0. \end{cases}
\end{aligned}
\tag{2.2.30}
$$

$$
\begin{aligned}
B_4(x) &:= -a_1 \int_{b_1}^{x} \left(\int_{-\infty}^{x_1} H_{b_1}(x_2) dx_2 \right) dx_1 \\
&= \begin{cases} -a_1 \left[\frac{x^2}{2} - b_1 x + \frac{b_1^2}{2} \right], & \text{if } \ x > b_1 \\ 0, & \text{if } \ x \leqslant b_1. \end{cases}
\end{aligned}
\tag{2.2.31}
$$

We first consider the case $x \to \infty$ for $F(\phi_0)(x)$. Summing up the equations (2.2.28)-(2.2.31), we get

$$
\begin{aligned}
F(\phi_0)(x) &= B_1(x) + B_2(x) + B_3(x) + B_4(x) \\
&= \sum_{j=0}^{k+2} \frac{(-1)^j}{(k+2-j)!j!} \{\overline{\mathcal{M}}_j(x) - \mathcal{M}_j\} x^{k+2-j} + \left(\frac{(-1)^{k-1}}{(k-1)!3!} \mathcal{M}_{k-1} - \frac{a_0}{3!} \right) x^3 \\
&\quad + \left(\frac{(-1)^k}{k!2} \mathcal{M}_k - \frac{a_1}{2} + \frac{a_0 b_0}{2} \right) x^2 + \left(\frac{(-1)^{k+1}}{(k+1)!} \mathcal{M}_{k+1} + a_1 b_1 - \frac{a_0 b_0^2}{2} \right) x \\
&\quad + \left(\frac{(-1)^{k+2}}{(k+2)!} \mathcal{M}_{k+2} + \frac{a_0 b_0^3}{3!} - \frac{a_1 b_1^2}{2} \right).
\end{aligned}
\tag{2.2.32}
$$

In view of the hypothesis (2.2.27), equations (2.2.10)-(2.2.13) can be written as

$$
\begin{aligned}
&\frac{(-1)^{k-1}}{(k-1)!3!} \mathcal{M}_{k-1} - \frac{a_0}{3!} = 0, \\
&\frac{(-1)^k}{k!2!} \mathcal{M}_k - \frac{a_1}{2} + \frac{a_0 b_0}{2} = 0, \\
&\frac{(-1)^{k+1}}{(k+1)!} \mathcal{M}_{k+1} + a_1 b_1 - \frac{a_0 b_0^2}{2} = 0, \\
&\frac{(-1)^{k+2}}{(k+2)!} \mathcal{M}_{k+2} + \frac{a_0 b_0^3}{3!} - \frac{a_1 b_1^2}{2} = 0.
\end{aligned}
$$

Thus, we are left with

$$
F(\phi_0)(x) = \sum_{j=0}^{k+2} \frac{(-1)^j}{(k+2-j)!j!} \{\overline{\mathcal{M}}_j(x) - \mathcal{M}_j\} x^{k+2-j}.
$$

For $0 < x \leq s < \infty$, we have

$$
x^{k+2-j} \leq s^{k+2-j} \text{ if } 0 \leq j \leq k+2. \tag{2.2.33}
$$

This leads to

$$x^{k+2-j}s^j|\phi_0(s)| \leq s^{k+2}|\phi_0(s)| \text{ if } 0 \leq j \leq k+2.$$

Integrating the above inequality with respect to s from x to ∞ and then dividing by $(k+2-j)!j!$, we arrive at

$$\frac{x^{k+2-j}}{(k+2-j)!j!}\int_x^\infty s^j|\phi_0(s)|ds \leq \frac{1}{(k+2-j)!j!}\int_x^\infty s^{k+2}|\phi_0(s)|ds. \qquad (2.2.34)$$

Thus,

$$\begin{aligned}\left|\sum_{j=0}^{k+2}\frac{(-1)^j}{(k+2-j)!j!}\left(\int_x^\infty s^j\phi_0(s)ds\right)x^{k+2-j}\right| &\leqslant \sum_{j=0}^{k+2}\frac{x^{k+2-j}}{(k+2-j)!j!}\int_x^\infty s^j|\phi_0(s)|ds \\ &\leqslant \sum_{j=0}^{k+2}\frac{1}{(k+2-j)!j!}\int_x^\infty s^{k+2}|\phi_0(s)|ds \\ &= o(|x|^{-1-\epsilon}) \quad \text{as} \quad x\to\infty,\end{aligned}$$

in view of $(1+|x|)^{k+3+\epsilon}\phi_0 \in L^1(\mathbb{R})$. Hence,

$$F(\phi_0)(x) = o(|x|^{-1-\epsilon}) \quad \text{as} \quad x\to\infty. \qquad (2.2.35)$$

Therefore, there exists a sufficiently large real number $L(>0)$ such that

$$|x^{1+\epsilon}F(\phi_0)(x)| < 1, \quad \forall x > L.$$

In othe words, $|F(\phi_0)(x)| < x^{-1-\epsilon} \quad \forall\, x > L$. Thus, we get

$$\int_L^\infty |F(\phi_0)(x)|dx < \int_L^\infty x^{-1-\epsilon}dx = \frac{1}{\epsilon L^\epsilon} < \infty. \qquad (2.2.36)$$

We now consider the case $x\to-\infty$ for $F(\phi_0)(x)$. If $x<0$, we have

$$\begin{aligned}\left|\sum_{j=0}^{k+2}\frac{(-1)^j}{(k+2-j)!j!}\left(\int_{-\infty}^x s^j\phi_0(s)ds\right)x^{k+2-j}\right| &\leqslant \sum_{j=0}^{k+2}\frac{|x|^{k+2-j}}{(k+2-j)!j!}\int_{-\infty}^x |s|^j|\phi_0(s)|ds \\ &\leqslant \sum_{j=0}^{k+2}\frac{1}{(k+2-j)!j!}\int_{-\infty}^x |s|^{k+2}|\phi_0(s)|ds \\ &= o(|x|^{-1-\epsilon}) \quad \text{as} \quad x\to-\infty. \quad (2.2.37)\end{aligned}$$

On the other hand, when x is smaller than minimum of b_0, b_1 and 0, it is to be noted from (2.2.29)-(2.2.31) that

$$B_2(x) = B_3(x) = B_4(x) = 0,$$

and thus from (2.2.32), we get

$$F(\phi_0)(x) \quad = \quad \sum_{j=0}^{k+2} \frac{(-1)^j}{(k+2-j)!j!} \overline{\mathcal{M}}_j(x) x^{k+2-j}. \tag{2.2.38}$$

Hence, in view of (2.2.37), we get

$$F(\phi_0)(x) = o(|x|^{-1-\epsilon}) \quad \text{as} \quad x \to -\infty. \tag{2.2.39}$$

Repeating as in the $x \to \infty$ case, one obtains that

$$\int_{-\infty}^{M} |F(\phi_0)(x)| dx < \infty \tag{2.2.40}$$

for some large number $|M|$ with $M < 0$.

Eventually, (2.2.35) and (2.2.39) will conclude that $F(\phi_0) \in L^1(\mathbb{R})$. □

Having $F(\phi_0)$ as L^1-function under the hypothesis of Lemma 2.2.3, we indicate its convolution with heat kernel $G_t(x)$ by

$$G_t * F(\phi_0)(x) = \int_{\mathbb{R}} G_t(x-y) F(\phi_0)(y) dy. \tag{2.2.41}$$

The following lemma tells that the difference term $\phi - \phi^k$ can be represented as the $(k+3)^{th}$ order spatial derivative of the convolution $G_t * F(\phi_0)$.

Lemma 2.2.4. *Assume that $\phi(x,t)$ is a solution to heat equation (2.1.1) subject to (2.1.2) with $(1+|x|)^{k+3+\epsilon}\phi_0 \in L^1(\mathbb{R})$, where k is any positive integer and ϵ is a small positive real number. Further, let*

$$\lim_{t \to 0^+} \int_{\mathbb{R}} x^j [\phi(x,t) - \phi^k(x,t)] dx = 0, \qquad 0 \leqslant j \leqslant k+2, \tag{2.2.42}$$

where ϕ^k is an asymptotic approximation given in (2.2.9). Then,

$$\partial_x^{k+3}(G_t * F(\phi_0)(x)) = \phi(x,t) - \phi^k(x,t) \quad for \quad t > 0, \ x \in \mathbb{R}. \tag{2.2.43}$$

Proof. Consider

$$\begin{aligned} &\partial_x^{k+3}(G_t * B_1(x)) \\ &= \int_{\mathbb{R}} G_t(x-y) \partial_y^{k+3} \left[\int_{-\infty}^{y} \left(\int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_{k+1}} \left(\int_{-\infty}^{x_{k+2}} \phi_0(x_{k+3}) dx_{k+3} \right) dx_{k+2} \cdots \right) dx_1 \right] dy \\ &= \phi(x,t). \end{aligned} \tag{2.2.44}$$

Similarly, we have

$$\begin{aligned}
\partial_x^{k+3}(G_t * B_2(x)) &= \partial_x^{j+1}(G_t * \partial_x^{k+2-j} B_2(x)) \\
&= -\sum_{j=0}^{k-2}(-1)^j \frac{\mathcal{M}_j}{j!}\partial_x^{j+1}\int_0^\infty G_t(x-y)dy \\
&= -\sum_{j=0}^{k-2}(-1)^j \frac{\mathcal{M}_j}{j!}\partial_x^{j} G_t(x). \qquad (2.2.45)
\end{aligned}$$

For $B_3(x)$, we compute that

$$\begin{aligned}
&\partial_x^{k+3}\int_{\mathbb{R}} G_t(x-y)B_3(y)dy \\
&= -a_0\partial_x^k\int_{\mathbb{R}} G_t(x-y)\partial_y^3\left[\int_{b_0}^{y}\left(\int_{b_0}^{x_1}\left(\int_{-\infty}^{x_2} H_{b_0}(x_3)dx_3\right)dx_2\right)dx_1\right]dy \\
&= -a_0\partial_x^k\int_{\mathbb{R}} G_t(x-y)H_{b_0}(y)dy \\
&= -a_0\partial_x^k\int_{-\infty}^{x} G_t(y-b_0)dy \\
&= -a_0\partial_x^{k-1}G_t(x-b_0). \qquad (2.2.46)
\end{aligned}$$

Similarly for $B_4(x)$, we obtain

$$\partial_x^{k+3}\int_{\mathbb{R}} G_t(x-y)B_4(y)dy = -a_1\partial_x^k G_t(x-b_1). \qquad (2.2.47)$$

Therefore, summing up the equations (2.2.44)-(2.2.47) lead to equation (2.2.43) as

$$F(\phi_0)(x) = B_1(x) + B_2(x) + B_3(x) + B_4(x).$$

□

Proof of Theorem 2.1.1. Using the Rodrigues formula for Hermite polynomials, it is seen that

$$\partial_x^{k+3}G_t(x) = \frac{(-1)^{k+3}}{\sqrt{\pi}(4t)^{\frac{k+4}{2}}}H_{k+3}(\xi)e^{-\xi^2}, \qquad (2.2.48)$$

where $\xi = x/\sqrt{4t}$ and H_{k+3} is a Hermite polynomial of degree $k+3$. Consider, for $1 \leqslant p \leqslant \infty$,

$$\begin{aligned}
\left\|\partial_x^{k+3}G_t(.)\right\|_{L^p(\mathbb{R})} &= \frac{1}{\sqrt{\pi}(4t)^{\frac{k+4}{2}}}\left(\sqrt{4t}\int_{-\infty}^{\infty}|H_{k+3}(\xi)e^{-\xi^2}|^p d\xi\right)^{1/p} \\
&= O(t^{\frac{1}{2p}-\frac{k+4}{2}}) \quad \text{as} \quad t\to\infty. \qquad (2.2.49)
\end{aligned}$$

Using Young's inequality, we find that

$$\left\|\partial_x^{k+3}(G_t * F(\phi_0))\right\|_{L^p(\mathbb{R})} \leqslant \|F(\phi_0)\|_{L^1(\mathbb{R})} \left\|\partial_x^{k+3} G_t\right\|_{L^p(\mathbb{R})}. \tag{2.2.50}$$

Therefore, in view of Lemma 2.2.3, Lemma 2.2.4 and the decay rate (2.2.49), the order of convergence in (2.1.6) follows from the Young's inequality (2.2.50). □

Repeating the proof of Theorem 2.1.1 with $(k+2)^{th}$ partial derivative of G_t in place of $(k+3)^{th}$ partial derivative, we obtain the following:

Remark 2.2.5. *Let $\phi(x,t)$ be a solution to heat equation (2.1.1) subject to (2.1.2) satisfying $(1+|x|)^{k+3+\epsilon}\phi_0 \in L^1(\mathbb{R})$. Then, for $1 \leqslant p \leqslant \infty$, we have*

$$\left\|\int_{-\infty}^{\cdot} [\phi(y,t) - \phi^k(y,t)]dy\right\|_{L^p(\mathbb{R})} = O(t^{-\frac{3+k}{2}+\frac{1}{2p}}), \quad t \to \infty, \tag{2.2.51}$$

where ϕ^k is given in (2.1.7).

Remark 2.2.6. *It is important to notice that assuming the hypothesis (2.2.27), we seek for expressions of a_0, a_1, b_1 and b_0 in equations (2.2.10)-(2.2.13). This, in turn, produces the conditional statement (2.1.7) from the proposed approximate solution (2.2.9). In other words, the moments, upto the order $(k+2)$, of $\phi(x,t)$ agree with those of $\phi^k(x,t)$ given in (2.1.7) as $t \to 0^+$.*

Remark 2.2.7. *It is known that (Rees, 1922; Irving, 2004) there exist at least two real solutions to the quartic equation (2.1.8) in all the cases except the following three cases:*

1. $a < 0,\ c > \frac{a^2}{4},\ \triangle > 0,$
2. $a \geq 0,\ \triangle > 0$
3. $\triangle = 0,\ c = \frac{a^2}{4}, b = 0,$

where $a = q - \dfrac{3p^2}{8}$, $b = r + \dfrac{p^3}{8} - \dfrac{pq}{2}$, $c = s - \dfrac{3p^4}{256} + \dfrac{p^2 q}{16} - \dfrac{pr}{4}$ *and*
$\triangle = 144ab^2c - 128a^2c^2 - 4a^3b^2 + 16a^4c - 27b^4 + 256c^3$
with
$$p = -\frac{4k(k+1)(k+2)M_{k-1}M_k}{k^2(k+1)(k+2)M_{k-1}^2},$$

$$q = \frac{12k(k+2)M_{k-1}M_{k+1}}{k^2(k+1)(k+2)M_{k-1}^2},$$
$$r = -\frac{24kM_{k-1}M_{k+2}}{k^2(k+1)(k+2)M_{k-1}^2},$$
$$s = \frac{12[2M_kM_{k+2}(k+1) - (k+2)M_{k+1}^2]}{k^2(k+1)^2(k+2)M_{k-1}^2}.$$

In case the coefficients of equation (2.1.8) satisfy one of those three cases, we donot have higher order approximate solutions in the proposed form (2.2.9) and the work is left for future research. However, if $M_{k-1}M_{k+1} = \frac{k+1}{2k}M_k^2$ *and* $\frac{6k^2}{(k+2)(k+1)}M_{k-1}^2M_{k+2} = M_k^3$ *in those three cases, we have higher order approximate solution given in the first statement of (2.1.7).*

Remark 2.2.8. *Let* $\mathcal{M}_{k-1} = 0$, $\mathcal{M}_k \neq 0$ *and* $\phi(x,t)$ *be a solution to heat equation (2.1.1) subject to (2.1.2) satisfying* $(1+|x|)^{k+2+\epsilon}\phi_0 \in L^1(\mathbb{R})$, *where* k *is any positive integer and* ϵ *is any small positive real number. Then, for* $1 \leqslant p \leqslant \infty$ *, we have*

$$||\phi(.,t) - \phi^k(.,t)||_{L^p(\mathbb{R})} = O(t^{-\frac{3+k}{2}+\frac{1}{2p}}), \quad t \to \infty, \tag{2.2.52}$$

where ϕ^k *is given by*

$$\phi^k(x,t) = \sum_{j=0}^{k-2}(-1)^j\frac{\mathcal{M}_j}{j!}\partial_x^jG_t(x) + (-1)^k\frac{\mathcal{M}_k}{k!}\partial_x^kG_t\left(x - \frac{\mathcal{M}_{k+1}}{(k+1)\mathcal{M}_k}\right). \tag{2.2.53}$$

2.2.1 On viscous Burgers Equation

We now extend these results to investigate the large time asymptotics for solutions of the viscous Burgers equation;

$$u_t + uu_x = u_{xx}, \quad x \in \mathbb{R},\ t > 0, \tag{2.2.54}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{2.2.55}$$

with the initial data u_0 satisfying

$$(1+|x|)^{k+3+\epsilon}u_0 \in L^1(\mathbb{R}), \tag{2.2.56}$$

for any positive integer k and a small real number $\epsilon > 0$.

Theorem 2.2.9. *Let* $u(x,t)$ *be a solution to the Burgers equation (2.2.54) subject to (2.2.55) with* u_0 *satisfying* $(1+|x|)^{k+3+\epsilon}u_0 \in L^1(\mathbb{R})$. *Let*

$$\mathcal{M}_r := \int_{\mathbb{R}} x^r\left[-\frac{1}{2}u_0(x)\exp\left(-\frac{1}{2}\int_{-\infty}^{x}u_0(y)dy\right)\right]dx, \quad r = 0,1,2,\ldots. \tag{2.2.57}$$

then there exists $T_k > 0$ *such that*

$$||u(.,t) - u^k(.,t)||_{L^p(\mathbb{R})} = O(t^{-\frac{4+k}{2}+\frac{1}{2p}}), \quad t \to \infty, \tag{2.2.58}$$

where $1 \leqslant p \leqslant \infty$ *and*

$$u^k(x,t) := -2\frac{\phi^k(x,t)}{1+\int_{-\infty}^{x}\phi^k(y,t)dy}, \qquad x \in \mathbb{R}, \quad t > 0, \tag{2.2.59}$$

is well defined for $x \in \mathbb{R},\ t \geqslant T_k$. *Here* ϕ_k *is given by (2.1.7).*

The proof of the theorem 2.2.9 is omitted as it is easily proved using the Theorem 2.1.1, Remark 2.2.5 and the standard arguments (Yanagisawa, 2007).

Remark 2.2.10. *Let* $\mathcal{M}_{k-1} = 0$ *and* $\mathcal{M}_k \neq 0$ *and* $u(x,t)$ *be a solution to the Burgers equation (2.2.54) subject to (2.2.55) with* u_0 *satisfying* $(1+|x|)^{k+2+\epsilon}u_0 \in L^1(\mathbb{R})$. *Then there exists* $T_k > 0$ *such that*

$$||u(.,t) - u^k(.,t)||_{L^p(\mathbb{R})} = O(t^{-\frac{3+k}{2}+\frac{1}{2p}}), \quad t \to \infty, \tag{2.2.60}$$

where $1 \leqslant p \leqslant \infty$ *and*

$$u^k(x,t) := -2\frac{\phi^k(x,t)}{1+\int_{-\infty}^{x}\phi^k(y,t)dy}, \qquad x \in \mathbb{R}, \quad t > 0, \tag{2.2.61}$$

is well defined for $x \in \mathbb{R},\ t \geqslant T_k$. *Here* ϕ_k *is given by (2.2.53).*

2.3 Comparison with Yanagisawa (2007)'s asymptotic approximate solution

We list out the advantages and disadvantages of the proposed asymptotic approximate solution (2.2.9) by comparing with Yanagisawa (2007)'s asymptotic approximate solution (2.1.5).

1. In the work of Yanagisawa (2007), if $\mathcal{M}_k \neq 0$ and $\mathcal{M}_{k+2}\mathcal{M}_k \leqslant \frac{k+2}{2(k+1)}[\mathcal{M}_{k+1}]^2$, then time shift $(t_k)_+ = 0$ and hence Yanagisawa's approximation (2.1.5) converges to the true solution ϕ of the problem (2.1.1)-(2.1.2) with an order of convergence $O(t^{-\frac{3+k}{2}+\frac{1}{2p}})$ as $t \to \infty$. Whereas the asymptotic approximate

solution (2.1.7) converges to the true solution ϕ of the problem (2.1.1)-(2.1.2) with an order of convergence $O(t^{-\frac{4+k}{2}+\frac{1}{2p}})$ as $t \to \infty$.

Following example explains the same for $k = 3$.

Example 2.3.1. *Consider a discontinuous initial data*

$$\phi_0(x) = \begin{cases} (-x)^{-7}, & -\infty < x < -\frac{1}{2}, \\ 0, & -\frac{1}{2} \leqslant x \leqslant 1, \\ x^{-7}, & 1 < x < \infty, \end{cases} \tag{2.3.62}$$

for the problem (2.1.1)-(2.1.2).

Firstly, it is to be observed that ϕ_0 has moments up to 5^{th} order only. They are

$$\mathcal{M}_0 = \frac{65}{6},\ \mathcal{M}_1 = \frac{-31}{5},\ \mathcal{M}_2 = \frac{17}{4},\ \mathcal{M}_3 = \frac{-7}{3},\ \mathcal{M}_4 = \frac{5}{2} \text{ and } \mathcal{M}_5 = -1.$$

It is easy to see that $\mathcal{M}_5\mathcal{M}_3 < \frac{5}{8}\mathcal{M}_4^2$.

Hence, we obtain

$$||\phi(.,t) - \phi^3(.,t)||_{L^p(\mathbb{R})} = O(t^{-\frac{6}{2}+\frac{1}{2p}}),\quad t \to \infty,$$

where $\phi^3(x,t)$ is an asymptotic approximation of Yanagisawa (2007). On the other hand, choosing

$$\begin{aligned} a_0 &= \frac{17}{8}, \\ b_0 &= \frac{1}{765}\left\{-140 + \sqrt{-37775 + 90\beta} + \sqrt{10\left[-7555 - 9\beta + 161509\sqrt{\frac{5}{-7555 + 18\beta}}\right]}\right\}, \end{aligned}$$

with $\beta = 17^{\frac{2}{3}} \times 333235^{\frac{1}{3}}$ and

$$\begin{aligned} a_1 &= \frac{7}{18} + \frac{17}{8}b_0, \\ b_1 &= \frac{3}{2}\left[\frac{51b_0^2 - 5}{153b_0 + 28}\right]. \end{aligned}$$

in asymptotic approximate solution (2.2.9), one gets

$$||\phi(.,t) - \phi^3(.,t)||_{L^p(\mathbb{R})} = O(t^{-\frac{7}{2}+\frac{1}{2p}}),\quad t \to \infty.$$

2. Suppose that $\mathcal{M}_k = 0$. We then have to look for the largest integer $s(< k)$ so that $\mathcal{M}_s \neq 0$ and so Yanagisawa (2007)'s approximation (2.1.5) converges to the true solution ϕ either with an order of convergence $O(t^{-\frac{4+s}{2}+\frac{1}{2p}})$ when $(t_k)_+ > 0$ or $O(t^{-\frac{3+s}{2}+\frac{1}{2p}})$ when $(t_k)_+ = 0$. However, it is possible to construct an approximation (2.2.9) which converges to true solution ϕ with an order of convergence $O(t^{-\frac{4+k}{2}+\frac{1}{2p}})$ as $t \to \infty$.

   Following example illustrates it with $k = 3$.

   Example 2.3.2. *Consider a discontinuous initial data defined by*

$$\phi_0(x) = \begin{cases} (-x)^{-7}, & -\infty < x < -1, \\ 0, & -1 \leqslant x \leqslant 1, \\ x^{-7}, & 1 < x < \infty, \end{cases} \tag{2.3.63}$$

   for the problem (2.1.1)-(2.1.2).

   It can be easily seen that $-\infty < \mathcal{M}_j < \infty$ for $0 \leqslant j \leqslant 5$ and $\mathcal{M}_j$ is not defined for $j \geqslant 6$. In particular, we find that

$$\mathcal{M}_0 = \frac{1}{3},\ \mathcal{M}_1 = 0,\ \mathcal{M}_2 = \frac{1}{2},\ \mathcal{M}_3 = 0,\ \mathcal{M}_4 = 1 \text{ and } \mathcal{M}_5 = 0. \tag{2.3.64}$$

   Further, notice that $\mathcal{M}_5\mathcal{M}_3 < \frac{5}{8}\mathcal{M}_4^2$. Therefore, we get

$$||\phi(.,t) - \phi^2(.,t)||_{L^p(\mathbb{R})} = O(t^{-\frac{5}{2}+\frac{1}{2p}}), \quad t \to \infty,$$

   where $\phi^2(x,t)$ is an asymptotic approximation (2.1.5) of Yanagisawa (2007). On the other hand, choosing

$$a_0 = \frac{1}{4},\ a_1 = \frac{b_0}{4},\ b_0 = \sqrt{-1 + \frac{2}{\sqrt{3}}},\ b_1 = \frac{3b_0^2 - 1}{6b_0},$$

   in the asymptotic approximate solution (2.2.9), one gets

$$||\phi(.,t) - \phi^3(.,t)||_{L^p(\mathbb{R})} = O(t^{-\frac{7}{2}+\frac{1}{2p}}), \quad t \to \infty.$$

3. If $\mathcal{M}_{k-1} = 0$, $\mathcal{M}_k \neq 0$ and $\mathcal{M}_{k+2}\mathcal{M}_k > \frac{k+2}{2(k+1)}[\mathcal{M}_{k+1}]^2$, then higher rate of convergence of the approximation to the true solution ϕ is obtained by considering the asymptotic approximate solution of Yanagisawa (2007) only, not the one in the form (2.2.9).

4. If $\mathcal{M}_{k-1} = 0$, $\mathcal{M}_k \neq 0$ and $\mathcal{M}_{k+2}\mathcal{M}_k < \frac{k+2}{2(k+1)}[\mathcal{M}_{k+1}]^2$, then Yanagisawa's approximate solution (2.1.5) as well as the proposed approximate solution (2.2.53) would give the same rate of convergence $\left[\text{i.e., } O\left(t^{-\frac{3+k}{2}+\frac{1}{2p}}\right) \text{ as } t \to \infty\right]$.

Chapter 3

Large Time Asymptotics with Error Estimates to Solutions of a Forced Burgers Equation

3.1 Introduction

Over the past decades, the study of the forced Burgers model

$$u_t + uu_x = \nu u_{xx} + f(x,t), \qquad x \in \mathbb{R},\ t > 0, \tag{3.1.1}$$

received much attention due to its promising applications in various fields of Science, Engineering and Biology (see, for example, (Xu et al., 2007) and the references therein). In this chapter, we are concerned with a forced Burgers equation, namely,

$$u_t + uu_x = \nu u_{xx} + \frac{\gamma}{(2\beta t + 1)^{3/2}}, \quad x \in \mathbb{R}, \quad t > 0, \tag{3.1.2}$$

where $\beta > 0$ and $\gamma \neq 0$ and ν is the viscosity coefficient, subject to

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{3.1.3}$$

where u_0 is continuous on $\mathbb{R}$ and $u_0 \in L^1(\mathbb{R})$.

The motivation to study the initial value problem (3.1.2)-(3.1.3) is as follows:

Consider the Cauchy problem for heat equation

$$\phi_t = \phi_{xx}, \ x \in \mathbb{R}, \ t > 0, \tag{3.1.4}$$

$$\phi(x,0) = \phi_0(x), \ x \in \mathbb{R}, \tag{3.1.5}$$

where ϕ_0 is either bounded almost everywhere and continuous on $\mathbb{R}$ or $\phi_0 \in L^1(\mathbb{R})$. Then, the solution of the Cauchy problem (3.1.4)-(3.1.5) is given by

$$\phi(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{(x-y)^2}{4t}} \phi_0(y)dy, \quad t > 0, \quad x \in \mathbb{R}. \tag{3.1.6}$$

Though the exact solution (3.1.6) of (3.1.4)-(3.1.5) is available, for most of the initial data (3.1.5), the evaluation of the integral involved in (3.1.6) is tedious. Hence one has to look for either numerics or asymptotic analysis. Further, the solution of the Burgers equation ((3.1.2) with $\gamma = 0$) subject to (3.1.3) is given by

$$u(x,t) = -2\nu \frac{\phi_x(x,t)}{\phi(x,t)}, \ x \in \mathbb{R}, \ t > 0, \tag{3.1.7}$$

where $\phi(x,t)$ is as in (3.1.6). In consideration of (3.1.6), one can imagine the difficulty in evaluating the integrals involved in numerator and denominator of (3.1.7). Two well known properties of the solution to the Burgers equation ((3.1.2) with $\gamma = 0$) subject to (3.1.3) are that the total mass is conserved and the solution decays uniformly to 0 as $t \to \infty$. In contrast to these two properties, the total mass of the solution of (3.1.2) subject to (3.1.3) is not conservative. In view of these facts, we expect that the asymptotic analysis is the better option to study (3.1.2)-(3.1.3).

Kloosterziel (1990) studied the large time asymptotic behavior of the diffusion equation (3.1.4) on infinite and semi-infinite domains, subject to the initial data ϕ_0 that is square summable with respect to the exponentially-growing weight function $\mathrm{e}^{\frac{1}{2}x^2}$. Witelski and Bernoff (1998) constructed self-similar solutions to (3.1.4) with a stronger initial data than (3.1.5) by incorporating mass, center of mass, and variance of the initial data into self-similar solutions. Miller and Bernoff (2003) estimated the rates of convergence to the self similar asymptotic approximation of the solution of Burgers equation ((3.1.2) with $\gamma = 0$). Hopf (1950) analyzed rigorously various properties of solutions including large time behavior and vanishing viscosity limit. For the study of asymptotics via separable solutions to a generalization of the Burgers equation, we refer to (Rao and Nath, 2015). Eule and Friedrich (2006) derived solution for an initial value problem of a special case of (3.1.1), namely,

$$u_t + uu_x = \nu u_{xx} + G(t)x, \quad x \in \mathbb{R}, \ t > 0, \tag{3.1.8}$$

by considering $G(t)$ as white noise force. Salas (2010) investigated (3.1.8) and constructed solutions via generalized Cole-Hopf transformation and travelling wave method. Ding et al. (2001) studied the asymptotic behavior of solutions to a forced Burgers equation, which is a special case of (3.1.8),

$$u_t + uu_x = \mu u_{xx} + kx, \quad x \in \mathbb{R}, \quad t > 0, \tag{3.1.9}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{3.1.10}$$

where $k > 0$ and the initial data $u_0(x)$ satisfies the conditions that $u_0(x) \in L^1_{loc}(\mathbb{R})$ and $\int^x u_0(r)dr = o(x^2)$ as $|x| \to \infty$. With the help of Cole-Hopf transformation (3.1.7), they reduced (3.1.9) to

$$\phi_t = \mu\phi_{xx} - \frac{x^2}{\mu}\phi, \quad x \in \mathbb{R}, \quad t > 0. \tag{3.1.11}$$

They expressed the solution of (3.1.11) as Fourier-Hermite series and eventually showed that the solution of the Cauchy problem (3.1.9)-(3.1.10) converges to ax as $t \to \infty$, where the constant a depends on k and the viscosity coefficient μ. Further, assuming the initial data

$$u_0(x) = o(x), \quad |x| \to \infty,$$

Ding and Ding (2003) showed that, for fixed (x, t), the solution of (3.1.9) converges to the weak solution of relevant inviscid forced Burgers equation as $\mu \to 0$. Rao and Yadav (2010a) constructed solutions to the forced Burgers equation (3.1.9) subject to the initial data containing bounded and compactly supported initial profiles. Rao and Yadav (2010c) obtained the solutions to a generalization of (3.1.9), namely,

$$u_t + uu_x = u_{xx} + \frac{kx}{(2\beta t + 1)^2}, \quad x \in \mathbb{R}, \quad t > 0. \tag{3.1.12}$$

They reduced (3.1.12) to the linear equation

$$\phi_t = \phi_{xx} - \frac{kx^2}{4(2\beta t + 1)^2}\phi, \quad x \in \mathbb{R}, \quad t > 0, \tag{3.1.13}$$

via Cole-Hopf transformation (3.1.7) and then expressed the solution of (3.1.13) in terms of self-similar solutions of concerned heat equation. This representation of solution for (3.1.13) revealed the large time behavior of the solution quickly. We refer to (Rao and Yadav, 2010b) for obtaining the large time asymptotics for solutions to (3.1.12) subject to a more general initial data. Recently, Yadav (2013) considered a system of forced Burgers equations and derived the large time behavior of the solutions.

In order to simplify our analysis, without loss of generality, we can scale space and time in (3.1.2) to obtain the forced Burgers equation

$$u_t + uu_x = u_{xx} + \frac{\gamma}{(2\beta t + 1)^{3/2}}, \quad x \in \mathbb{R}, \quad t > 0, \tag{3.1.14}$$

where $\beta > 0$ and γ is a non-zero constant. We study (3.1.14) supplemented with an initial function

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}, \tag{3.1.15}$$

where $u_0(x)$ is continuous on $\mathbb{R}$ and $u_0 \in L^1(\mathbb{R})$. Firstly, we construct solutions to the forced Burgers equation (3.1.14) subject to the initial data

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}, \tag{3.1.16}$$

with the condition

$$\int_{-\infty}^{\infty} \mathrm{e}^{\frac{\beta}{2}x^2 - \int_0^x u_0(r)dr} dx < \infty. \tag{3.1.17}$$

For which, we seek similarity solutions for (3.1.14) and then employ Cole-Hopf like transformation to the resulting nonlinear partial differential equation. We then make use of the method of separation of variables to obtain solutions of the concerned partial differential equation. This process yields solutions explicitly in terms of Hermite polynomials. We then prove the existence of solution to the forced Burgers equation (3.1.14) subject to the initial condition (3.1.15). Moreover, we will give an asymptotic approximate solution to the Cauchy problem (3.1.14)-(3.1.15) with a uniform error of order $O(t^{-\frac{1}{2}})$ for large time.

This chapter is organized as follows. Section 3.2 deals with the construction of solutions to the forced Burgers equation (3.1.14) subject to the initial data (3.1.16)-(3.1.17). In section 3.3, we prove the existence of solution to the Cauchy problem (3.1.14)-(3.1.15) and then give an approximate solution for it with an error estimation.

3.2 Solutions in terms of Hermite polynomials

In this section, we construct explicit solutions to the initial value problem (3.1.14), (3.1.16)-(3.1.17). We introduce similarity variables

$$t, \quad \eta = \frac{x}{b(t)}, \quad u(x,t) = \frac{1}{c(t)}\phi(\eta, t), \tag{3.2.18}$$

where $b(t)$ and $c(t)$ are to be chosen in such a way that the forced Burgers equation (3.1.14) is transformed to a partial differential equation which is suitable for applying the method of separation of variables. Making use of these variables (3.2.18) in (3.1.14), it can be seen that $\phi(\eta, t)$ solves

$$\begin{aligned} b^2(t)\phi_t - b(t)\left[b'(t)\eta\phi_\eta + \frac{c'(t)}{c(t)}b(t)\phi\right] + \frac{b(t)}{c(t)}\phi\phi_\eta = \\ \phi_{\eta\eta} + b^2(t)c(t)\frac{\gamma}{(2\beta t+1)^{3/2}}. \end{aligned} \tag{3.2.19}$$

Assumption of

$$b(t) = a_1 c(t), \quad a_1 \text{ is any nonzero real number,}$$

simplifies (3.2.19) to

$$b^2(t)\phi_t - b(t)b'(t)\left[\eta\phi_\eta + \phi\right] + a_1\phi\phi_\eta = \phi_{\eta\eta} + \frac{\gamma}{a_1}\frac{b^3(t)}{(2\beta t+1)^{3/2}}. \tag{3.2.20}$$

Let

$$\Phi(\eta, t) = \int_0^\eta \phi(r, t)dr,$$

so that

$$\phi(\eta, t) = \Phi_\eta(\eta, t). \tag{3.2.21}$$

When (3.2.21) is substituted into (3.2.20), we obtain that

$$\frac{\partial}{\partial \eta}\left[b^2(t)\Phi_t - b(t)b'(t)\eta\Phi_\eta + \frac{a_1}{2}\Phi_\eta^2 - \Phi_{\eta\eta} - \frac{\gamma}{a_1}\frac{b^3(t)\eta}{(2\beta t+1)^{3/2}}\right] = 0.$$

Therefore, it is enough to look for a smooth function $\Phi(\eta, t)$ such that

$$b^2(t)\Phi_t - b(t)b'(t)\eta\Phi_\eta + \frac{a_1}{2}\Phi_\eta^2 - \Phi_{\eta\eta} - \frac{\gamma}{a_1}\frac{b^3(t)\eta}{(2\beta t+1)^{3/2}} = 0. \qquad (3.2.22)$$

Set

$$\Phi(\eta, t) = a_2 \log \psi(\eta, t),$$

where a_2 is any non-zero arbitrary constant. Then, one can see that ψ satisfies the partial differential equation

$$\psi\left[b^2(t)\psi_t - b(t)b'(t)\eta\psi_\eta - \psi_{\eta\eta} - \frac{\gamma}{a_1 a_2}\frac{b^3(t)\eta}{(2\beta t+1)^{3/2}}\psi\right] = -\left(\frac{a_1 a_2}{2} + 1\right)\psi_\eta^2.$$

Choosing $a_2 = -2/a_1$, the above partial differential equation is reduced to

$$b^2(t)\psi_t - b(t)b'(t)\eta\psi_\eta - \psi_{\eta\eta} + \frac{\gamma}{2}\frac{b^3(t)\eta}{(2\beta t+1)^{3/2}}\psi = 0. \qquad (3.2.23)$$

It is seen that the assumption of $b(t)b'(t) = \beta$; i.e.,

$$b(t) = \sqrt{2\beta t + 1}, \qquad (3.2.24)$$

makes equation (3.2.23) amenable for the method of separation of variables. Making use of (3.2.24) into (3.2.23), we are left with

$$(2\beta t + 1)\psi_t = \beta\eta\psi_\eta + \psi_{\eta\eta} - \frac{\gamma}{2}\eta\psi. \qquad (3.2.25)$$

Summing up the transformations used to reduce the forced Burgers equation (3.1.14) to the linear partial differential equation (3.2.25), we arrive at

$$u(x, t) = \frac{-2}{\sqrt{2\beta t + 1}}\frac{\psi_\eta(\eta, t)}{\psi(\eta, t)}, \qquad (3.2.26)$$

where $\eta = \frac{x}{\sqrt{2\beta t+1}}$ and call it as Cole-Hopf like transformation.

One may suspect here whether the solution ψ of (3.2.25) would imply the solution of (3.1.14) via Cole-Hopf like transformation (3.2.26) as we took specifically zero in right hand side of (3.2.22) rather than an arbitrary function of single variable t. In Appendix of this chapter, it will be shown that this reverse process also holds.

The equation (3.2.25) amounts to looking for solutions of the form

$$\psi(\eta, t) = G(\eta)T(t). \tag{3.2.27}$$

When this expression is substituted in (3.2.25), the result can be written as

$$\frac{G''(\eta) + \beta\eta G'(\eta) - \frac{\gamma}{2}\eta G(\eta)}{G(\eta)} = \frac{(2\beta t + 1)T'(t)}{T(t)}. \tag{3.2.28}$$

Assuming μ is any separation constant, equation (3.2.28) splits into two ordinary differential equations for $G(\eta)$ and $T(t)$:

$$G''(\eta) + \beta\eta G'(\eta) - \left[\mu + \frac{\gamma}{2}\eta\right] G(\eta) = 0, \tag{3.2.29}$$

$$T'(t) - \frac{\mu}{2\beta t + 1} T(t) = 0. \tag{3.2.30}$$

We now reduce the differential equation (3.2.29) into normal form. For which, we introduce $\tilde{G}(\eta)$ as follows:

$$\tilde{G}(\eta) = \mathrm{e}^{\beta\eta^2/4} G(\eta). \tag{3.2.31}$$

Substituting (3.2.31) into (3.2.29), one has

$$\tilde{G}''(\eta) - \left[\frac{\beta^2}{4}\eta^2 + \frac{\gamma}{2}\eta + \frac{\beta}{2} + \mu\right] \tilde{G}(\eta) = 0. \tag{3.2.32}$$

We introduce new variables (Polyanin and Zaitsev, 2003)

$$\xi = \eta + \frac{\gamma}{\beta^2}, \qquad \hat{G}(\xi) = \tilde{G}(\eta).$$

It then follows that the equation (3.2.32) is transformed to

$$\hat{G}''(\xi) - \left[\frac{\beta^2}{4}\xi^2 + \left(\mu + \frac{\beta}{2} - \frac{\gamma^2}{4\beta^2}\right)\right] \hat{G}(\xi) = 0. \tag{3.2.33}$$

For the eigen values

$$\mu_n = \frac{\gamma^2}{4\beta^2} - (n+1)\beta, \quad n = 0, 1, \ldots, \tag{3.2.34}$$

the eigen functions $G_n(\xi)$ of (3.2.33) are given by

$$G_n(\xi) = \frac{1}{a_n} H_n\left(\sqrt{\frac{\beta}{2}}\xi\right) \mathrm{e}^{-\frac{\beta}{4}\xi^2}, \tag{3.2.35}$$

where $a_n = \sqrt{2^n n! \sqrt{\frac{2\pi}{\beta}}}$. Here H_n's are nth degree Hermite polynomials given by

$$H_n(x) = (-1)^n \mathrm{e}^{x^2} \frac{d^n}{dx^n}(\mathrm{e}^{-x^2}).$$

It can be seen that the sequence $\{G_n(\xi)\}$ satisfies the orthonormal property

$$\int_{-\infty}^{\infty} G_n(\xi) G_m(\xi) d\xi = \delta_{nm},$$

where δ_{mn} is the Kronecker's delta. Further it is known that $\{G_n(\xi)\}$ is complete in $L^2(\mathbb{R})$ (see (Higgins, 1977)). Solving (3.2.30) for $T(t)$ with (3.2.34), one has

$$T_n(t) = (2\beta t + 1)^{\frac{\gamma^2}{8\beta^3} - \frac{n+1}{2}}, \quad n = 0, 1, 2, \dots. \tag{3.2.36}$$

Thus, the resulting products of the form (3.2.27) lead to

$$\psi(\eta, t) = \sum_{n=0}^{\infty} c_n T_n(t) G_n\left(\eta + \frac{\gamma}{\beta^2}\right) \exp(-\beta\eta^2/4), \tag{3.2.37}$$

where $T_n(t)$ and $G_n(\xi)$ are given by (3.2.36) and (3.2.35) respectively. We now temporarily assume that the sum of the above series in the case $t = 0$ is $\psi(\eta, 0)$. That is, the equation (3.2.37) reduces to

$$\psi(\eta, 0)\mathrm{e}^{\frac{\beta}{4}\eta^2} = \sum_{n=0}^{\infty} c_n G_n\left(\eta + \frac{\gamma}{\beta^2}\right),$$

for $t = 0$. This makes sense if $\psi(\xi - \frac{\gamma}{\beta^2}, 0)\mathrm{e}^{\frac{\beta}{4}(\xi - \frac{\gamma}{\beta^2})^2}$ is square summable over $\mathbb{R}$ with respect to the variable ξ as the sequence $\{G_n(\xi)\}$ is complete in $L^2(\mathbb{R})$. Our assumption on u_0 in (3.1.17) ensures that $\psi(\eta, 0)\mathrm{e}^{\frac{\beta}{4}\eta^2} \in L^2(\mathbb{R})$. It follows that the numbers c_n's are computed by

$$\begin{aligned} c_n &= \int_{-\infty}^{\infty} \psi(\xi - \gamma/\beta^2, 0)\mathrm{e}^{\frac{\beta(\xi - \gamma/\beta^2)^2}{4}} G_n(\xi) d\xi, \qquad n = 0, 1, 2, \dots \\ &= \frac{1}{\sqrt{n! 2^n \sqrt{2\pi/\beta}}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}\int^{\eta} u_0(r)dr - \frac{\gamma}{2\beta}\eta - \frac{\gamma^2}{4\beta^3}} H_n\left(\sqrt{\frac{\beta}{2}}\left(\eta + \frac{\gamma}{\beta^2}\right)\right) d\eta. \end{aligned} \tag{3.2.38}$$

For future reference, we study the large time asymptotics to the solutions of the linear partial differential equation (3.2.25). For $n = 0$, a simplification in (3.2.38) gives that

$$\begin{aligned} c_0 &= \left(\frac{\beta}{2\pi}\right)^{\frac{1}{4}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}\int^{\eta} u_0(r)dr - \frac{\gamma}{2\beta}\eta - \frac{\gamma^2}{4\beta^3}} d\eta \\ &\neq 0. \end{aligned}$$

Hence, (3.2.37) would imply that

$$\begin{aligned} \psi(\eta,t) &\sim c_0 T_0(t)\exp(-\beta\eta^2/4) G_0\left(\eta + \frac{\gamma}{\beta^2}\right), \ \eta = O(1) \text{ as } t \to \infty \\ &= \left(\frac{\beta}{2\pi}\right)^{\frac{1}{4}} c_0 (2\beta t + 1)^{\frac{\gamma^2}{8\beta^3} - \frac{1}{2}} \mathrm{e}^{-\left[\frac{\beta}{2}\eta^2 + \frac{\gamma}{2\beta}\eta + \frac{\gamma^2}{4\beta^3}\right]}, \ \eta = O(1) \text{ as } t \to \infty. \end{aligned}$$

From this asymptotic behavior, one may conclude the following when η is bounded:

- $\psi(\eta,t)$ diverges to ∞ as $t \to \infty$ when $4\beta^3 < \gamma^2$,
- $\psi(\eta,t)$ decays to 0 as $t \to \infty$ when $4\beta^3 > \gamma^2$, (3.2.39)
- $\psi(\eta,t)$ converges to a non zero constant for the rest case.

Eventually, the solution of (3.1.14), for all $t > 0$ and $x \in \mathbb{R}$, is derived as follows:

$$\begin{aligned} u(x,t) &= \frac{-2}{\sqrt{2\beta t + 1}} \frac{\psi_\eta(\eta,t)}{\psi(\eta,t)} \\ &= -\frac{2\sum_{n=0}^{\infty} \frac{c_n}{\sqrt{n!2^n}} (2\beta t + 1)^{\frac{\gamma^2}{8\beta^3} - \frac{n+1}{2}} \partial_\eta \left(\widetilde{H}_n(\eta)\right)}{\sum_{n=0}^{\infty} \frac{c_n}{\sqrt{n!2^n}} (2\beta t + 1)^{\frac{\gamma^2}{8\beta^3} - \frac{n}{2}} \widetilde{H}_n(\eta)}, \end{aligned} \quad (3.2.40)$$

where $\eta = \frac{x}{\sqrt{2\beta t+1}}$, $\widetilde{H}_n(\eta) = H_n\left(\sqrt{\frac{\beta}{2}}\left(\eta + \frac{\gamma}{\beta^2}\right)\right) e^{-\left[\frac{\beta}{2}\eta^2 + \frac{\gamma}{2\beta}\eta + \frac{\gamma^2}{4\beta^3}\right]}$ and the coefficients c_n are given by (3.2.38).

3.3 Large time asymptotics with error estimates

In this section, we prove the existence of a solution for the forced Burgers equation (3.1.14) subject to the initial data (3.1.15). Then we give an approximate solution for the initial value problem (3.1.14)-(3.1.15) with an error of order $O(t^{-\frac{1}{2}})$ in L^p-norm, where $1 \leqslant p \leqslant \infty$, for large time. For which, we first prove the existence of a solution to the linear partial differential equation (3.2.25) subject to the initial condition

$$\psi(\eta, 0) = \exp\left\{-\frac{1}{2}\int_0^\eta u_0(r)dr\right\} =: \psi_0(\eta), \tag{3.3.41}$$

where $u_0 \in L^1(\mathbb{R})$ and is continuous on $\mathbb{R}$. Substituting (3.2.36), (3.2.38) and (3.3.41) in (3.2.37), we get

$$\psi(\eta, t) = \frac{(2\beta t+1)^{\frac{\gamma^2}{8\beta^3}}}{\sqrt{2\beta t+1}} \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \widehat{G}(\eta, y, t) \mathrm{e}^{\frac{\beta}{4}(y^2-\eta^2)} \psi_0(y) dy, \tag{3.3.42}$$

where

$$\widehat{G}(\eta, y, t) = \frac{1}{(2\beta t+1)^{\frac{n}{2}}} G_n\left(\eta + \frac{\gamma}{\beta^2}\right) G_n\left(y + \frac{\gamma}{\beta^2}\right).$$

It is to be noted that the above expression for $\psi(\eta, t)$ was derived by considering the initial data (3.1.16)-(3.1.17) of (3.1.14), but not (3.1.15) of (3.1.14).

We now replace the series in (3.3.42) by its sum and hence represent $\psi(\eta, t)$ only in the integral form. Then we show that it is a classical solution of the linear partial differential equation (3.2.25) for $t > 0$ and satisfies the initial condition (3.3.41) in the limit sense $t \to 0$. For which let us recall the following result from (Titchmarsh, 1986):

$$\sum_{n=0}^{\infty} \frac{\mathrm{e}^{-\frac{1}{2}(x^2+y^2)}}{2^n n!} t^n H_n(x) H_n(y) = \frac{1}{\sqrt{1-t^2}} \exp\left\{\frac{x^2-y^2}{2} - \frac{(x-yt)^2}{1-t^2}\right\},$$

if $| t |< 1$. Thus, since $\left|\frac{1}{\sqrt{2\beta t+1}}\right| < 1$ when $t > 0$, it is deduced that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{(2\beta t+1)^{\frac{n}{2}}} G_n\left(\eta + \frac{\gamma}{\beta^2}\right) G_n\left(y + \frac{\gamma}{\beta^2}\right) &= \sqrt{\frac{2\beta t+1}{4\pi t}} \\ \times \exp\left\{\frac{\beta}{4}\left[\left(\eta + \frac{\gamma}{\beta^2}\right)^2 - \left(y + \frac{\gamma}{\beta^2}\right)^2\right] - \frac{1}{4t}\left[P(\eta, y, t)\right]^2\right\}&, \end{aligned} \tag{3.3.43}$$

where

$$P(\eta, y, t) = \sqrt{2\beta t+1}\left(\eta + \frac{\gamma}{\beta^2}\right) - \left(y + \frac{\gamma}{\beta^2}\right). \tag{3.3.44}$$

Using (3.3.43) in (3.3.42), we get

$$\psi(\eta, t) = (2\beta t+1)^{\frac{\gamma^2}{8\beta^3}} \int_{-\infty}^{\infty} K(\eta, y, t) \psi_0(y) dy, \tag{3.3.45}$$

where

$$K(\eta, y, t) = \frac{1}{2\sqrt{\pi t}} \exp\left\{\frac{\gamma}{2\beta}(\eta - y) - \frac{1}{4t}[P(\eta, y, t)]^2\right\}, \qquad (3.3.46)$$

and P is given in (3.3.44).

The following lemma says that the kernel $K(\eta, y, t)$ in (3.3.46) behaves like that of heat kernel as $t \to 0$.

Lemma 3.3.1. *Suppose that* $\int_{-\infty}^{\infty} |u_0(x)|dx < \infty$ *and* u_0 *is continuous on* $\mathbb{R}$. *Then*

$$\lim_{t\to 0} \int_{-\infty}^{\infty} K(\eta, y, t)\psi_0(y)dy = \psi_0(\eta),$$

where $\psi_0(\eta)$ *and* $K(\eta, y, t)$ *are given in (3.3.41) and (3.3.46) respectively.*

Proof. Set

$$s := \frac{\gamma}{\beta^2} - \sqrt{2\beta t + 1}\left(\eta + \frac{\gamma}{\beta^2}\right) + \frac{\gamma t}{\beta}. \qquad (3.3.47)$$

In view of (3.3.46) and (3.3.47), we have

$$\int_{-\infty}^{\infty} K(\eta, y, t)dy = \frac{1}{2\sqrt{\pi t}} e^{\frac{\gamma}{2\beta}\left[\eta + s - \frac{\gamma}{2\beta}t\right]} \int_{-\infty}^{\infty} e^{-\frac{1}{4t}(y+s)^2} dy. \qquad (3.3.48)$$

If we substitute $y = v - s$ in equation (3.3.48), we find that

$$\begin{aligned} \int_{-\infty}^{\infty} K(\eta, y, t)dy &= \frac{1}{2\sqrt{\pi t}} e^{\frac{\gamma}{2\beta}\left[\eta + s - \frac{\gamma}{2\beta}t\right]} \int_{-\infty}^{\infty} e^{-\frac{1}{4t}v^2} dv \\ &= e^{\frac{\gamma}{2\beta}\left[\eta + s - \frac{\gamma}{2\beta}t\right]} \to 1, \quad \text{as} \quad t \to 0. \end{aligned}$$

Hence,

$$\lim_{t\to 0} \int_{-\infty}^{\infty} K(\eta, y, t)\psi_0(y)dy - \psi_0(\eta) = \lim_{t\to 0} \int_{-\infty}^{\infty} K(\eta, y, t)[\psi_0(y) - \psi_0(\eta)]dy.$$

Once again changing the variable y as above, we get

$$\begin{aligned} I &:= \int_{-\infty}^{\infty} K(\eta, y, t)[\psi_0(y) - \psi_0(\eta)]dy \\ &= \frac{1}{2\sqrt{\pi t}} e^{\frac{\gamma}{2\beta}\left[\eta + s - \frac{\gamma}{2\beta}t\right]} \int_{-\infty}^{\infty} e^{-\frac{1}{4t}v^2}[\psi_0(v - s) - \psi_0(\eta)]dv \\ &= \frac{1}{\sqrt{\pi}} e^{\frac{\gamma}{2\beta}\left[\eta + s - \frac{\gamma}{2\beta}t\right]} \int_{-\infty}^{\infty} e^{-z^2}[\psi_0(2\sqrt{t}z - s) - \psi_0(\eta)]dz. \end{aligned}$$

Let $\epsilon > 0$ be given and $\eta \in \mathbb{R}$ be fixed. Take any (small) real number $\delta_1 > 0$ and then define c as follows:

$$c \equiv \sup_{\{t:|t|<\delta_1\}} \left\{\frac{1}{\sqrt{\pi}} e^{\frac{\gamma}{2\beta}\left[\eta + s - \frac{\gamma}{2\beta}t\right]}\right\}. \qquad (3.3.49)$$

As $\int_{-\infty}^{\infty} |u_0(z)|dz < \infty$, we have

$$\begin{aligned} |\psi_0(2\sqrt{t}z - s)| &= \left| \exp\left\{ -\frac{1}{2} \int_0^{2\sqrt{t}z-s} u_0(r)dr \right\} \right| \\ &\leqslant \mathrm{e}^{\frac{\|u_0\|_1}{2}}. \end{aligned}$$

Hence,

$$\int_A \mathrm{e}^{-z^2} |\psi_0(2\sqrt{t}z - s)|dz \leqslant \mathrm{e}^{\frac{\|u_0\|_1}{2}} \int_A \mathrm{e}^{-z^2} dz, \tag{3.3.50}$$

for any subset A of $\mathbb{R}$. We now choose sufficiently large numbers $L_1 > 0$ and $L_2 > 0$ such that

$$\int_{|z|>L_1} \mathrm{e}^{-z^2} dz < \frac{\epsilon}{4c\ \mathrm{e}^{\frac{\|u_0\|_1}{2}}}, \tag{3.3.51}$$

$$\int_{|z|>L_2} \mathrm{e}^{-z^2} dz < \frac{\epsilon}{4c\ \mathrm{e}^{\frac{\|u_0\|_1}{2}}}. \tag{3.3.52}$$

Assume L to be the maximum of the numbers L_1 and L_2. Then, in veiw of (3.3.49)-(3.3.52), one gets

$$\begin{aligned} &\left| \frac{1}{\sqrt{\pi}} \mathrm{e}^{\frac{\gamma}{2\beta}[\eta + s - \frac{\gamma}{2\beta}t]} \int_{|z|>L} \mathrm{e}^{-z^2} [\psi_0(2\sqrt{t}z - s) - \psi_0(\eta)]dz \right| \\ &\leqslant c \left| \int_{|z|>L} \mathrm{e}^{-z^2} \psi_0(2\sqrt{t}z - s)dz \right| + c|\psi_0(\eta)| \left| \int_{|z|>L} \mathrm{e}^{-z^2} dz \right| \\ &\leqslant \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}, \quad \text{whenever } |t| < \delta_1. \end{aligned} \tag{3.3.53}$$

As u_0 is continuous in $\mathbb{R}$, ψ_0 is uniformly continuous on any closed interval containing the set

$$\left\{ 2\sqrt{t}z + \sqrt{2\beta t + 1}\left(\eta + \frac{\gamma}{\beta^2}\right) - \frac{\gamma}{\beta^2} - \frac{\gamma t}{\beta} : |z| < L \quad \text{and} \quad |t| < \delta_1 \right\}.$$

This assures that there exists a small number $\delta_2 > 0$ such that

$$|\psi_0(2\sqrt{t}z - s) - \psi_0(\eta)| < \frac{\epsilon}{2\sqrt{\pi}c} \quad \text{whenever} \quad |2\sqrt{t}z - s - \eta| < \delta_2,$$

where c is given in (3.3.49). Let $g_t(z) := 2\sqrt{t}z - s - \eta$, $\forall\ z \in [-L, L]$, where s is given in (3.3.47). It is seen that $g_t(z)$ converges uniformly to 0 as $t \to 0$. Hence there exists a small number $\delta(\leqslant \delta_1) > 0$ such that $|g_t(z)| < \delta_2$ whenever $|t| < \delta$. Thus, we obtain

$$\left| \frac{1}{\sqrt{\pi}} \mathrm{e}^{\frac{\gamma}{2\beta}[\eta + s - \frac{\gamma}{2\beta}t]} \int_{-L}^{L} \mathrm{e}^{-z^2} [\psi_0(2\sqrt{t}z - s) - \psi_0(\eta)]dz \right| < \frac{\epsilon}{2}, \tag{3.3.54}$$

whenever $|t| < \delta$.

Finally, in view of (3.3.53) and (3.3.54), the modulus of I is smaller than ϵ when $|t| < \delta$. This completes the proof. $\square$

Theorem 3.3.2. *Suppose that* $\int_{-\infty}^{\infty} |u_0(x)|dx < \infty$ *and* u_0 *is continuous on* $\mathbb{R}$. *Then the integral representation*

$$(2\beta t + 1)^{\frac{\gamma^2}{8\beta^3}} \int_{-\infty}^{\infty} K(\eta, y, t)\psi_0(y)dy = \psi(\eta, t), \tag{3.3.55}$$

becomes a classical solution of the partial differential equation (3.2.25) *satisfying the initial data in the limit sense as follows:*

$$\lim_{t\to 0} \psi(\eta, t) = \psi_0(\eta), \tag{3.3.56}$$

where $\psi_0(\eta)$ *and* $K(\eta, y, t)$ *are as in (3.3.41) and (3.3.46) respectively.*

Proof. The assumption $u_0 \in L^1(\mathbb{R})$ implies that $\psi_0(\eta) \leqslant \mathrm{e}^{\frac{\|u_0\|_1}{2}}$, which in turn implies that the integral

$$(2\beta t + 1)^{\frac{\gamma^2}{8\beta^3}} \int_{-\infty}^{\infty} K(\eta, y, t)\psi_0(y)dy,$$

converges for all $\eta \in \mathbb{R}$ and $t > 0$.

Let

$$\begin{aligned} W(\eta, y, t) &:= (2\beta t + 1)^{\frac{\gamma^2}{8\beta^3}} K(\eta, y, t) \\ &=: \frac{(2\beta t + 1)^{\frac{\gamma^2}{8\beta^3}}}{2\sqrt{\pi t}} J(\eta, y, t), \end{aligned} \tag{3.3.57}$$

where

$$J(\eta, y, t) = \exp\left\{\frac{\gamma}{2\beta}(\eta - y) - \frac{1}{4t}[P(\eta, y, t)]^2\right\},$$

and $P(\eta, y, t)$ is given in (3.3.44). To show that $\psi(\eta, t)$ satisfies equation (3.2.25), it suffices to verify that

$$(2\beta t + 1)W_t = \beta\eta W_\eta + W_{\eta\eta} - \frac{\gamma}{2}\eta W.$$

Computing partial derivatives for W, we obtain

$$\begin{aligned} W_t = &\frac{(2\beta t + 1)^{\frac{\gamma^2}{8\beta^3}}}{2\sqrt{\pi t}} J(\eta, y, t) \\ &\times \left\{\frac{\gamma^2}{4\beta^2(2\beta t + 1)} - \frac{1}{2t} - \frac{\beta}{2t\sqrt{2\beta t + 1}}\left(\eta + \frac{\gamma}{\beta^2}\right)P + \frac{1}{4t^2}P^2\right\}, \end{aligned} \tag{3.3.58}$$

$$W_\eta = \frac{(2\beta t+1)^{\frac{\gamma^2}{8\beta^3}}}{2\sqrt{\pi t}} J(\eta,y,t)\left\{\frac{\gamma}{2\beta} - \frac{\sqrt{2\beta t+1}}{2t}P\right\}, \qquad (3.3.59)$$

and

$$W_{\eta\eta} = \frac{(2\beta t+1)^{\frac{\gamma^2}{8\beta^3}}}{2\sqrt{\pi t}} J(\eta,y,t)\left\{\left[\frac{\gamma}{2\beta} - \frac{\sqrt{2\beta t+1}}{2t}P\right]^2 - \frac{2\beta t+1}{2t}\right\}. \qquad (3.3.60)$$

From (3.3.57) – (3.3.60), one can see that

$$\begin{aligned}\beta\eta W_\eta + W_{\eta\eta} - \frac{\gamma}{2}\eta W &= \frac{(2\beta t+1)^{\frac{\gamma^2}{8\beta^3}}}{2\sqrt{\pi t}} J(\eta,y,t) \\ &\times\left\{\frac{\gamma^2}{4\beta^2} - \frac{2\beta t+1}{2t} - \frac{\beta\sqrt{2\beta t+1}}{2t}\left(\eta+\frac{\gamma}{\beta^2}\right)P + \frac{2\beta t+1}{4t^2}P^2\right\} \\ &= (2\beta t+1)W_t.\end{aligned}$$

Hence $\psi(\eta,t)$ satisfies the partial differential equation (3.2.25) and proof of (3.3.56) follows from the Lemma 3.3.1. □

Proposition 3.3.3. *Suppose that the initial data $u_0(x)$ satisfies either the condition $\exp\{-\frac{1}{2}\int_0^x u_0(r)dr\} \in L^2(\mathbb{R}, e^{\beta x^2/2})$ or $u_0(x) = o(x)$ for large $|x|$ and u_0 is continuous on $\mathbb{R}$. Then also the integral representation (3.3.55) converges for all $\eta \in \mathbb{R}$, $t > 0$ and satisfies the linear partial differential equation (3.2.25).*

Proof. Suppose that $u_0(x)$ satisfies the condition

$$\exp\{-\frac{1}{2}\int_0^x u_0(r)dr\} \in L^2(\mathbb{R}, e^{\beta x^2/2}).$$

Then

$$e^{-\frac{1}{2}\int_0^x u_0(r)dr + \frac{\beta x^2}{4}} \in L^2(\mathbb{R})$$

and we know that $e^{-\frac{\beta x^2}{4}} \in L^2(\mathbb{R})$ as $\beta > 0$. Hence, using the Holder's inequality, we have

$$\begin{aligned}\int_{\mathbb{R}} e^{-\frac{1}{2}\int_0^x u_0(r)dr}dx &= \int_{\mathbb{R}} e^{-\frac{1}{2}\int_0^x u_0(r)dr + \frac{\beta x^2}{4} - \frac{\beta x^2}{4}}dx \\ &\le \left\|e^{-\frac{1}{2}\int_0^x u_0(r)dr + \frac{\beta x^2}{4}}\right\|_2 \left\|e^{-\frac{\beta x^2}{4}}\right\|_2 \\ &< \infty. \qquad (3.3.61)\end{aligned}$$

From this, it is clear that $u_0(x)$ satisfies

$$\exp\left\{-\int_0^x u_0(r)dr\right\} \in L^1(\mathbb{R}). \tag{3.3.62}$$

Now using the inequality of arithmetic and geometric means, we can find that

$$|K(\eta, y, t)\psi_0(y)| \leqslant \frac{1}{2}\left[K^2(\eta, y, t) + \psi_0^2(y)\right].$$

Integrating the above inequality over $(-\infty, \infty)$ with respect to y, we get

$$\begin{aligned}\int_{-\infty}^{\infty} |K(\eta, y, t)\psi_0(y)|dy &\leqslant \frac{1}{2}\int_{-\infty}^{\infty} K^2(\eta, y, t)dy + \frac{1}{2}\int_{-\infty}^{\infty} \psi_0^2(y)dy \\ &\leqslant \frac{1}{8\pi t}\mathrm{e}^{\frac{\gamma}{\beta}\left[\eta+s-\frac{\gamma}{2\beta}t\right]}\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2t}v^2}dv \\ &+ \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{-\int_0^y u_0(r)dr}dy,\end{aligned} \tag{3.3.63}$$

where $v = y + s$ and $s = \frac{\gamma}{\beta^2} - \sqrt{2\beta t + 1}\left(\eta + \frac{\gamma}{\beta^2}\right) + \frac{\gamma t}{\beta}$. In view of (3.3.62) and (3.3.63), we can conclude that $\psi(\eta, t)$ converges for all $\eta \in \mathbb{R}$ and $t > 0$.

The assumptions $u_0(x) = o(x)$ for large $|x|$ and u_0 is continuous on $\mathbb{R}$ imply that $\psi_0(\eta) = \mathrm{e}^{o(\eta^2)}$ for large $|\eta|$ and ψ_0 is continuous on $\mathbb{R}$, these two facts assure that integral representation of $\psi(\eta, t)$ in (3.3.55) converges for all $\eta \in \mathbb{R}$ and $t > 0$. □

Remark 3.3.4. *The function classes*

$$\{f(x) : f(x) = o(x) \text{ for large } |x| \text{ and } f \text{ is continuous on } \mathbb{R}\}$$

and

$$\left\{f(x) : \exp\{-\frac{1}{2}\int_0^x f(r)dr\} \in L^2(\mathbb{R}, \mathrm{e}^{\beta x^2/2})\right\}$$

are disjoint.

Theorem 3.3.5. *Suppose that* $\int_{-\infty}^{\infty} |u_0(x)|dx < \infty$ *and* u_0 *is continuous in* $\mathbb{R}$. *Then*

$$u(x, t) = \frac{-2}{\sqrt{2\beta t + 1}} \frac{\int_{-\infty}^{\infty} \frac{\partial}{\partial\eta}K(\eta, y, t)\psi_0(y)dy}{\int_{-\infty}^{\infty} K(\eta, y, t)\psi_0(y)dy} \quad \forall\ x \in \mathbb{R},\ t > 0, \tag{3.3.64}$$

defines a solution of (3.1.14) *satisfying the initial data in the following sense:*

$$\int_0^x u(r, t)dr \to \int_0^x u_0(r)dr \quad \text{as} \quad t \to 0, \tag{3.3.65}$$

where $\eta = \frac{x}{\sqrt{2\beta t+1}}$. *And there exists a positive real number* C *such that*

$$\left| u(x,t) - \frac{\gamma}{\beta}\left(1 - \frac{1}{\sqrt{2\beta t+1}}\right)\right| \leqslant C\frac{1}{\sqrt{t}}, \ \forall\ x \in \mathbb{R} \text{ and } t > 0, \tag{3.3.66}$$

where $\psi_0(\eta)$ *and* $K(\eta, y, t)$ *are as in (3.3.41) and (3.3.46) respectively.*

Proof. Since $u_0 \in L^1(\mathbb{R})$, it is clear that the integral

$$\begin{aligned} \psi(\eta,t) &= (2\beta t+1)^{\frac{\gamma^2}{8\beta^3}} \int_{-\infty}^{\infty} K(\eta,y,t)\psi_0(y)dy \\ &= \frac{(2\beta t+1)^{\frac{\gamma^2}{8\beta^3}}}{\sqrt{\pi}} \mathrm{e}^{\frac{\gamma}{2\beta}\left[\eta+s-\frac{\gamma}{2\beta}t\right]} \int_{-\infty}^{\infty} \mathrm{e}^{-z^2}\psi_0(2\sqrt{t}z-s)dz, \end{aligned} \tag{3.3.67}$$

where $s = \frac{\gamma}{\beta^2} - \sqrt{2\beta t+1}\left(\eta + \frac{\gamma}{\beta^2}\right) + \frac{\gamma t}{\beta}$, converges for all $\eta \in \mathbb{R}$ and $t > 0$. From the Theorem 3.3.2, we conclude that $\psi(\eta,t)$ solves (3.2.25) and satisfies the condition

$$\lim_{t\to 0} \psi(\eta,t) = \psi_0(\eta).$$

Hence

$$u(x,t) = \frac{-2}{\sqrt{2\beta t+1}} \frac{\psi_\eta(\eta,t)}{\psi(\eta,t)},$$

defines a solution of (3.1.14) for $t > 0$. Further, since $\eta = \dfrac{x}{\sqrt{2\beta t+1}}$, we have the following for $F(\eta,t) \equiv \frac{\psi_\eta(\eta,t)}{\psi(\eta,t)}$:

$$\begin{aligned} \int_0^x u(r,t)dr &= -\frac{2}{\sqrt{1+2\beta t}} \int_0^x F\left(\frac{r}{\sqrt{2\beta t+1}}, t\right) dr \\ &= -2\int_0^\eta F(s,t)ds \\ &= -2\log\psi(\eta,t) + 2\log\psi(0,t) \\ &\to -2\log\psi_0(x) + 2\log\psi_0(0) \\ &= -2\int_0^x \frac{\psi_{0\eta}(\eta)}{\psi_0(\eta)} d\eta \\ &= \int_0^x u_0(r)dr \quad \text{as} \quad t \to 0. \end{aligned}$$

Thus, we are left to prove the inequality (3.3.66). Finding partial derivative of ψ in equation (3.3.67) with respect to η, we have

$$\begin{aligned} \psi_\eta(\eta,t) &= \frac{(2\beta t+1)^{\frac{\gamma^2}{8\beta^3}}}{\sqrt{\pi}} \mathrm{e}^{\frac{\gamma}{2\beta}\left[\eta+s-\frac{\gamma}{2\beta}t\right]} \\ &\times \int_{-\infty}^{\infty} \left[\frac{\gamma}{2\beta}\left(1-\sqrt{2\beta t+1}\right) + \frac{\sqrt{2\beta t+1}}{\sqrt{t}} z\right] \mathrm{e}^{-z^2}\psi_0(2\sqrt{t}z-s)dz. \end{aligned}$$

Therefore,

$$
\begin{aligned}
u(x,t) &= \frac{-2}{\sqrt{2\beta t+1}}\frac{\psi_\eta(\eta,t)}{\psi(\eta,t)} \\
&= \frac{\gamma}{\beta}\left(1-\frac{1}{\sqrt{2\beta t+1}}\right)-\frac{2}{\sqrt{t}}\frac{\int_{-\infty}^{\infty} z\mathrm{e}^{-z^2}\psi_0(2\sqrt{t}z-s)dz}{\int_{-\infty}^{\infty}\mathrm{e}^{-z^2}\psi_0(2\sqrt{t}z-s)dz}. \quad (3.3.68)
\end{aligned}
$$

We know that

$$
-\int_0^{2\sqrt{t}z-s} u_0(r)dr \leqslant \|u_0\|_1
$$

and so

$$
\mathrm{e}^{-\frac{1}{2}\int_0^{2\sqrt{t}z-s}u_0(r)dr} \leqslant \mathrm{e}^{\frac{\|u_0\|_1}{2}}.
$$

Hence,

$$
\begin{aligned}
\left|\int_{-\infty}^{\infty} z\mathrm{e}^{-z^2}\psi_0(2\sqrt{t}z-s)dz\right| &\leqslant \int_{-\infty}^{\infty}|z|\mathrm{e}^{-z^2}|\psi_0(2\sqrt{t}z-s)|dz \\
&\leqslant \mathrm{e}^{\frac{\|u_0\|_1}{2}}\int_{-\infty}^{\infty}|z|\mathrm{e}^{-z^2}dz = \mathrm{e}^{\frac{\|u_0\|_1}{2}}, \quad (3.3.69)
\end{aligned}
$$

where the result $\int_{-\infty}^{\infty}|z|\mathrm{e}^{-z^2}dz = 1$ is used. One can also take

$$
\int_0^{2\sqrt{t}z-s} u_0(r)dr \leqslant \|u_0\|_1
$$

and so

$$
\mathrm{e}^{-\frac{1}{2}\int_0^{2\sqrt{t}z-s}u_0(r)dr} \geqslant \mathrm{e}^{-\frac{\|u_0\|_1}{2}}.
$$

Multiplying the above inequality by e^{-z^2} and integrating over $\mathbb{R}$, we obtain

$$
\int_{-\infty}^{\infty}\mathrm{e}^{-z^2}\psi_0(2\sqrt{t}z-s)dz \geqslant \sqrt{\pi}\mathrm{e}^{-\frac{\|u_0\|_1}{2}} > 0
$$

and hence

$$
\left|\int_{-\infty}^{\infty}\mathrm{e}^{-z^2}\psi_0(2\sqrt{t}z-s)dz\right| \geqslant \sqrt{\pi}\mathrm{e}^{-\frac{\|u_0\|_1}{2}}. \quad (3.3.70)
$$

Clearly (3.3.66) follows from (3.3.68) in view of the inequalities (3.3.69) and (3.3.70) with $C = \frac{2}{\sqrt{\pi}}\mathrm{e}^{\|u_0\|_1}$. □

Remark 3.3.6. *Suppose that $u_0(x) \in L^1_{loc}(\mathbb{R})$, u_0 is continuous on $\mathbb{R}$ and $u_0(x) = o(x)$ for large $|x|$. Then also the statement of (3.3.64) and (3.3.65) of the Theorem 3.3.5 holds by virtue of the Proposition 3.3.3.*

Having proved the inequality (3.3.66), we are ready to give the error estimates. Set

$$v_0(x,t) := \frac{\gamma}{\beta} - \frac{\gamma}{\beta\sqrt{2\beta t+1}}.$$

It is to be observed that $v_0(x,t)$ is a solution of the initial value problem (3.1.14)-(3.1.15) with a specific initial data $u_0(x) \equiv 0 \ \forall \ x \in \mathbb{R}$. Let us now treat $u(x,t)$ as a (true) solution and $v_0(x,t)$ as an approximate solution of the initial value problem (3.1.14)-(3.1.15). Then we show that the approximate solution $v_0(x,t)$ differs from the true solution $u(x,t)$ by an error of the order $O(t^{-\frac{1}{2}})$ as $t \to \infty$ with respect to the L^p-norm, where $1 \leqslant p \leqslant \infty$. In view of the inequality (3.3.66) of the Theorem 3.3.5, $v_0(x,t)$ converges uniformly to $u(x,t)$ as $t \to \infty$. Hence

$$\|u(.,t) - v_0(.,t)\|_{L^\infty(\mathbb{R})} \to 0 \quad \text{as} \quad t \to \infty.$$

Further, we can say that

$$\|u(.,t) - v_0(.,t)\|_{L^\infty(\mathbb{R})} = O(t^{-\frac{1}{2}}) \quad \text{as} \quad t \to \infty.$$

If $\mathcal{K}$ is any compact subset of $\mathbb{R}$, then

$$\|u(.,t) - v_0(.,t)\|_{L^p(\mathcal{K})} = O(t^{-\frac{1}{2}}) \quad \text{as} \quad t \to \infty,$$

where $1 \leqslant p < \infty$.

Remark 3.3.7. *It is to be noted that $u(x,t)$ is convergent uniformly as $t \to \infty$ irrespective of $\psi(\eta,t)$ in (3.2.40) is convergent or divergent, in view of the asymptotic behavior of $\psi(\eta,t)$ discussed in (3.2.39) and (3.3.66).*

Remark 3.3.8. *We now mention two more solutions of the forced Burgers equation (3.1.14),*

1. $v_1(x,t) := \frac{2\beta x}{2\beta t+1} + \frac{\gamma}{\beta\sqrt{2\beta t+1}}, \ \forall \ x \in \mathbb{R}, \ t > 0,$
2. $v_2(x,t) := \frac{\gamma}{\beta\sqrt{2\beta t+1}}\left(1 + \frac{1}{\beta t}\right) + \frac{x}{2\beta t+1}\left(2\beta + \frac{1}{t}\right), \ \forall \ x \in \mathbb{R}, \ t > 0.$

In fact, the solutions $v_1(x,t)$ and $v_2(x,t)$ were noticed while studying the large time behavior of the solution $\psi(\eta,t)$ of the partial differential equation (3.2.25) as well as its partial derivative $\psi_\eta(\eta,t)$ and while studying the expression of $u(x,t)$ in (3.3.68) respectively.

3.4 Appendix

As mentioned in Section 3.2, we now show that the expression $u(x,t)$ obtained through the Cole-Hopf like transformation

$$u(x,t) = \frac{-2}{\sqrt{2\beta t+1}} \frac{\psi_\eta(\eta,t)}{\psi(\eta,t)}, \tag{3.4.71}$$

where $\eta = \frac{x}{\sqrt{2\beta t+1}}$, would satisfy the forced Burgers equation (3.1.14) if $\psi(\eta,t)$ is obtained by solving (3.2.25).

Suppose that the solution $\psi(\eta,t)$ of (3.2.25) is found. Then introduce $\Phi(\eta,t)$ as follows:

$$\Phi(\eta,t) \equiv -2\log\left(\psi(\eta,t)\right), \qquad \eta \in \mathbb{R},\ t > 0. \tag{3.4.72}$$

Hence, substitution of $\psi(\eta,t) = \mathrm{e}^{-\frac{1}{2}\Phi(\eta,t)}$ into (3.2.25) gives the partial differential equation:

$$(1+2\beta t)\Phi_t = \beta\eta\Phi_\eta + \Phi_{\eta\eta} - \frac{1}{2}(\Phi_\eta)^2 + \gamma\eta. \tag{3.4.73}$$

Differentiating the above equation with respect to η, one obtains

$$(1+2\beta t)\Phi_{\eta t} = \beta(\eta\Phi_{\eta\eta} + \Phi_\eta) + \Phi_{\eta\eta\eta} - \Phi_\eta\Phi_{\eta\eta} + \gamma. \tag{3.4.74}$$

We now pick up the spacial derivative of Φ as follows:

$$\phi(\eta,t) \equiv \Phi_\eta(\eta,t). \tag{3.4.75}$$

This expression will reduce the partial differential equation (3.4.74) to

$$(1+2\beta t)\phi_t = \beta(\eta\phi_\eta + \phi) + \phi_{\eta\eta} - \phi\phi_\eta + \gamma. \tag{3.4.76}$$

Let us also introduce a dependent and independent variables as follows:

$$\begin{aligned} v(x,t) &\equiv \phi(\eta,t), \\ x &= \sqrt{1+2\beta t}\,\eta, \quad t = t. \end{aligned} \tag{3.4.77}$$

With these variables, the equation (3.4.76) is transformed to

$$(1+2\beta t)v_t = \beta v + (1+2\beta t)v_{xx} - \sqrt{1+2\beta t}\ vv_x + \gamma. \tag{3.4.78}$$

Finally, introduce

$$u(x,t) \equiv \frac{v(x,t)}{\sqrt{1+2\beta t}}. \tag{3.4.79}$$

Summing up the transformations (3.4.79), (3.4.77), (3.4.75) and (3.4.72) gives us the same Cole-Hopf like transformation (3.4.71). In view of (3.4.79), the equation (3.4.78) changes to the forced Burgers equation (3.1.14) concluding the required result.

Chapter 4

Conclusion

An asymptotic approximate solution to the Cauchy problem for diffusion equation (2.1.1)-(2.1.2) was proposed in Chapter 2 and its higher order convergence was proved. This approximate solution was constructed by making the first $k+3$ initial moments of it to agree with those of the true solution. Large time asymptotics for diffusion equation as well as Burgers equation were studied with the help of proposed approximate solution. Finally, some advantages of the proposed approximation were given by comparing it with Yanagisawa (2007)'s approximate solution.

In Chapter 3, explicit solutions in terms of Hermite polynomials were derived to the viscous Burgers equation with a forcing term (3.1.14) subject to the initial data (3.1.16)-(3.1.17). This process mainly depended on the Cole-Hopf like transformation and then the method of separation of variables. Further, we proved the existence of a solution to Cauchy problem (3.1.14)-(3.1.15). We found an approximate solution to (3.1.14)-(3.1.15) which is, in fact, the solution to (3.1.14)-(3.1.15) with $u_0(x)$ is identically 0 on $\mathbb{R}$. Finally, we established the rates of convergence to asymptotic approximation as order $O(t^{-\frac{1}{2}})$ in L^p-norm, where $1 \leqslant p \leqslant \infty$, for large time. However, more challenging task is to obtain higher order error estimates for the problem considered in Chapter 3. One may ask whether the asymptotic approximation given in Chapter 3 is unique or not with the decay rate $O(t^{-\frac{1}{2}})$ for large time. We have no answer for this question right now.

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BIODATA

Name	:	Mohd Ahmed
Email	:	ahmed.nitw@gmail.com
Date of Birth	:	06 February 1989.
Permanent address	:	Mohd Ahmed, *S/o* Mohd Pasha, H. No: 1-10-94/B/3, Sha sahab gutta, Mahabubnagar, Telangana-509 001.

Educational Qualifications :

Degree	**Year**	**Institution / University**
B.Sc. Mathematics	2011	Adarsh degree college, Mahabubnagar. (Osmania University)
M.Sc. Applied Mathematics	2013	NIT-Warangal.