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Strongly indexable graphs and applications

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ABSTRACT

In 1990, Acharya and Hegde introduced the concept of strongly k-indexable graphs: A (p, q)-graph $G = (V, E)$ is said to be *strongly k-indexable* if its vertices can be assigned distinct numbers $0, 1, 2, \ldots, p-1$ so that the values of the edges, obtained as the sums of the numbers assigned to their end vertices form an arithmetic progression $k, k+1, k+2, \ldots, k+(q-1)$. When $k = 1$, a strongly k-indexable graph is simply called a strongly indexable graph. In this paper, we report some results on strongly k-indexable graphs and give an application of strongly k-indexable graphs to *plane geometry*, viz; *construction of polygons of same internal angles and sides of distinct lengths*.

1. Introduction

For all terminology and notation in graph theory, we follow Harary [7] and West [12].

Graph labelings, where the *vertices* and *edges* are assigned *real values* or *subsets of a set* subject to certain conditions, have often been motivated by their utility in various applied fields and their intrinsic mathematical interest (logico-mathematical). Graph labelings were first introduced in the mid sixties. In the intervening years, dozens of graph labeling problems have been studied in over six hundred papers. An enormous body of literature has grown around the subject, especially in the last forty years or so, and is still getting embellished due to an increasing number of application driven concepts [6].

Labeled graphs are becoming an increasingly useful family of *mathematical models* for a broad range of applications. The qualitative labelings of graph elements have inspired research in diverse fields of human enquiry such as *conflict resolution in social psychology*, *electrical circuit theory*and *energy crisis*. Quantitative labelings of graphs have led to quite intricate fields of applications such as *Coding Theory problems*, including the design of good *radar location codes, synch-set codes, missile guidance codes* and *convolution codes with optimal autocorrelation properties*. Labeled graphs have also been applied in determining ambiguities in *X-Ray Crystallographic analysis*, to design *communication network addressing systems*, to determine *optimal circuit layouts and radio-astronomy*, etc.

In this section, we mention necessary definitions and some important results on additive theme based labelings [1,2, 8–10] etc.

Given a (p, q)-graph $G = (V, E)$, the set $\mathcal{N}$ of nonnegative integers, a finite subset $\mathcal{A}$ of $\mathcal{N}$ and a commutative binary operation $+ : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$, every vertex function $f : V(G) \to \mathcal{A}$ induces an edge function $f^+ : E(G) \to \mathcal{N}$ such that $f^+(uv) = f(u) + f(v)\ \forall\ uv \in E(G)$. Such vertex functions are called *additive vertex functions* [1,2].

Theorem 1.1 ([1]). *For any graph $G = (V, E)$ and for any vertex function $f : V(G) \to \mathcal{N}$*

$$\sum_{e \in E(G)} f^+(e) = \sum_{v \in V(G)} d(v) f(v). \tag{1}$$

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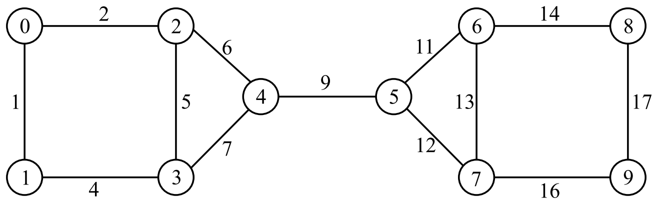

Fig. 1.1. An indexable graph.

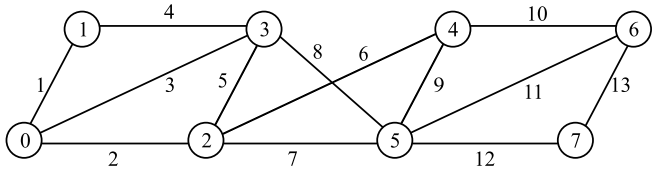

Fig. 1.2. A strongly indexed graph.

Definition 1.2 ([1]). An *additive labeling* of a graph G is an injective vertex function f such that induced edge function f^+ is injective.

For the given (p, q)-graph $G = (V, E)$ define

1. $f(G) = \{f(u) : u \in V(G)\}$.
2. $f^+(G) = \{f^+(e) : e \in E(G)\}$.

Acharya and Hegde have introduced the concept of (k, d)-arithmetic and strongly (k, d)-indexable graphs.

Definition 1.3 ([1]). An additive labeling f of a graph G is said to be a (k, d)-*arithmetic labeling* (*arithmetic labeling*) where k and d are positive integers if $f^+(G) = \{k, k+d, k+2d, \ldots, k+(q-1)d\}$. A graph which admits such a labeling for at least one pair of values of k and d is called (k, d)-arithmetic (or arithmetic) graph.

Definition 1.4 ([1]). An additive labeling f of a graph G is said to be an *indexable labeling* if $f : V(G) \to \{0, 1, 2, \ldots, p-1\}$ such that the values in $f^+(G)$ are all distinct. A graph which admits such a labeling is called an *indexable graph*.

An example of an indexable graph is displayed in Fig. 1.1.

Definition 1.5 ([1,8]). An indexable labeling of a graph G with $f^+(G) = \{k, k+d, \ldots, k+(q-1)d\}$ is called *strongly* (k, d)-*indexable labeling* of G.

Definition 1.6 ([1,8,3]). A strongly (k, d)-indexable labeling of a (p, q) graph G with $d = 1$ is called a *strongly k-indexable labeling*. A graph which admits such a labeling for at least one value of k is called *strongly k-indexable graph*.

Strongly 1-indexable graphs are simply called *strongly indexable* or *strongly indexed graphs*. An example of a strongly indexed graph is given in Fig. 1.2.

Kotzig and Rosa [11] have introduced the concept of edge-magic graphs.

Definition 1.7. A graph G is said to be edge-magic if it admits a bijection $f : V \cup E \to \{1, 2, \ldots, p+q\}$ such that $f(u) + f(v) + f(uv) = c(f)$, a constant for all $uv \in E$.

Enomoto et al., [4] have introduced the concept of super edge-magic graph.

Definition 1.8. A graph G is said to be super edge-magic if it admits a bijection $f : V \cup E \to \{1, 2, \ldots, p+q\}$ with $f(V) = \{1, 2, \ldots, p\}$ and $f(E) = \{p+1, p+2, \ldots, p+q\}$ such that $f(u) + f(v) + f(uv) = c(f)$, a constant for all $uv \in E$.

Remark 1.9. From the above definition, one can prove that, if a graph G is strongly k-indexable, then it is super edge magic with $c(f) = p+q+k+2$. Also it can be proved that if a graph G is super edge magic then it is strongly k-indexable for $k = c(f) - p - q - 2$. One can verify that the cycle C_5 is super edge magic with $c(f) = 14$ and strongly 2-indexable. But C_5 is not strongly indexable (see Theorem 2, Acharya and Hegde [3]). Thus, it is clear that every strongly k-indexable graph is super edge magic irrespective of k but every super edge magic graph is not strongly k-indexable for all k especially when $k = 1$. So the study of strongly k-indexable graphs is more useful.

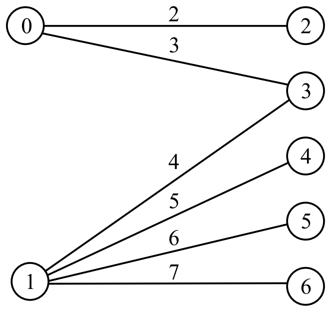

Fig. 2.1. A 2-indexed $C_{2,5}$.

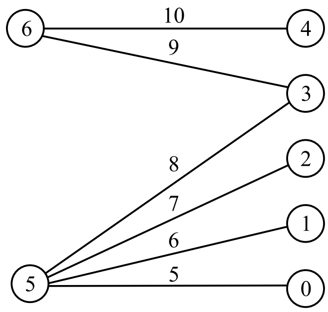

Fig. 2.2. A 5-indexed $C_{2,5}$.

2. Strongly *k*-indexable graphs

In this section, we investigate finite strongly k-indexable graphs. The next result gives an idea of how to generate a strongly k-indexable labeling from a given strongly k-indexable labeling of the graph G.

Theorem 2.1. *A (p, q)-graph $G = (V, E)$ is strongly k-indexable if and only if it is strongly $[(2p - q - 1) - k]$-indexable.*

Proof. Let $G = (V, E)$ be a (p, q)-graph which is strongly k-indexable with strongly k-indexable labeling f.

Then define a labeling $g : V(G) \to \{0, 1, 2, \ldots, p - 1\}$ by $g(v_i) = (p - 1) - f(v_i)$ for all $v_i \in V(G)$.

Note that

$$g^+(uv) = 2(p - 1) - (f(u) + f(v)) = 2p - 2 - f^+(uv).$$

Therefore clearly g^+ is injective. Since $f^+(G) = \{k + i | 0 \le i \le q - 1\}$, we get

$$\begin{aligned} g^+(G) &= \{2p - 2 - (k + i) | 0 \le i \le q - 1\} \\ &= \{2p - 2 - (k + (q - 1 - j)) | 0 \le j \le q - 1\} \\ &= \{2p - q - 1 - k + j | 0 \le j \le q - 1\}. \end{aligned}$$

Hence g is strongly $[(2p - q - 1) - k]$-indexable labeling of G. Similarly, one can prove the converse. ◇

Corollary 2.2. *If a (p, q)-graph is strongly k-indexable, then $1 \le k \le 2p - q - 2$.*

Corollary 2.3. *A tree T on p vertices is strongly k-indexable if and only if it is strongly $(p - k)$-indexable; in particular, if T is strongly k-indexable, then $1 \le k \le p - 1$.*

Definition 2.4 (*[6]*). A *caterpillar* is a tree, the deletion of whose pendant vertices results in a path.

For example, strongly k-indexable labeling and strongly $[(2p - q - 1) - k]$-indexable labeling of Caterpillar $C_{2,5}$ using the above theorem are displayed in Figs. 2.1 and 2.2, respectively (Note that $k = 2$).

It is important to note that the sets of labels in the strongly k-indexable labeling and strongly $[(2p-q-1)-k]$-indexable labeling need not be distinct. For example, strongly k-indexable labeling and strongly $(2p - q - 1 - k)$-indexable labelings of Petersen graph using Theorem 2.1, are displayed in Fig. 2.3. (Note that $k = (2p - q - 1) - k$ when $k = 2$ for Petersen graph).

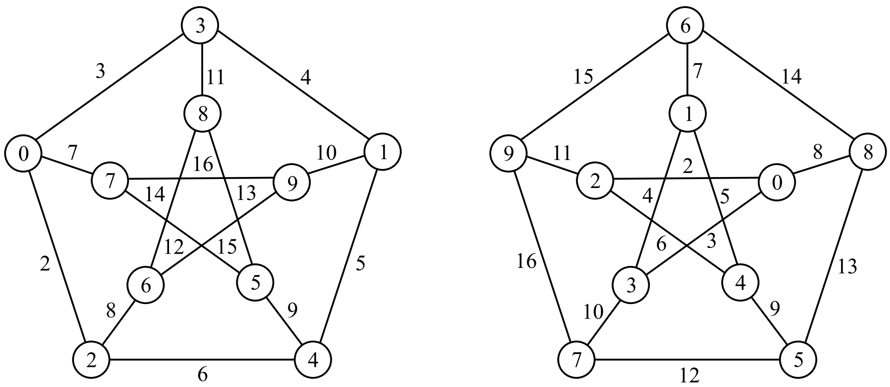

Fig. 2.3. A Strongly 2-indexed and $[(2p-q-1)-2]$-indexed Petersen graph.

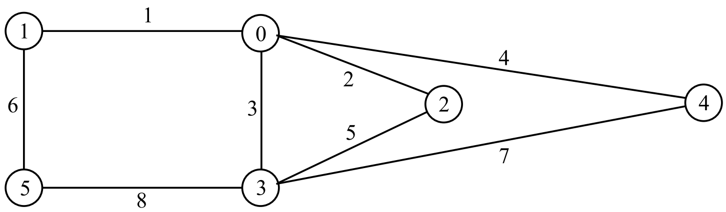

Fig. 2.4. Strongly 1-indexed eulerian graph.

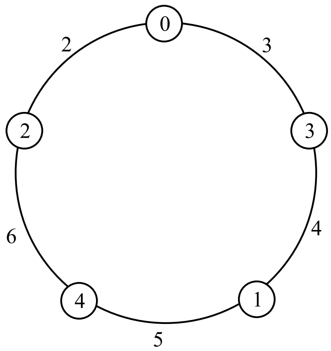

Fig. 2.5. Strongly 2-indexed C_5.

The following result gives a necessary condition for an eulerian graph to be strongly k-indexable.

Theorem 2.5. *Let G be an eulerian (p, q)-graph that is strongly k-indexable.*

1. *If k is odd then $q \equiv 0, 3 (\mathrm{mod}\ 4)$.*
2. *If k is even then $q \equiv 0, 1 (\mathrm{mod}\ 4)$.*

Proof. Let $G = (V, E)$ be a strongly k-indexable (p, q)-eulerian graph with a strongly k-indexable labeling f. From Eq. (1) we get

$$\sum_{u \in V} \deg(u) f(u) = qk + \frac{q(q-1)}{2}. \tag{2}$$

Case 1: Let $k = 2m + 1$

Since the degree of every vertex of an eulerian graph is even, the left side of the Eq. (2) is even and the right side must be even. This is possible only when $q \equiv 0, 3 (\mathrm{mod}\ 4)$.

Case 2: Let $k = 2m$

Similarly, to get an even number on the right side of (2) we must have, $q \equiv 0, 1 (\mathrm{mod}\ 4)$.

For example, in the cases $q \equiv 0 (\mathrm{mod}\ 4)$, $q \equiv 1 (\mathrm{mod}\ 4)$ and $q \equiv 3 (\mathrm{mod}\ 4)$ respectively, strongly 1-indexable, strongly 2-indexable and strongly 3-indexable labeling of eulerian graphs are displayed in Figs. 2.4–2.6.

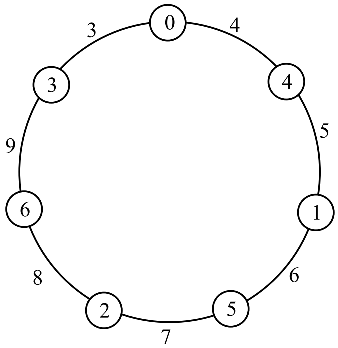

Fig. 2.6. Strongly 3-indexed C_7.

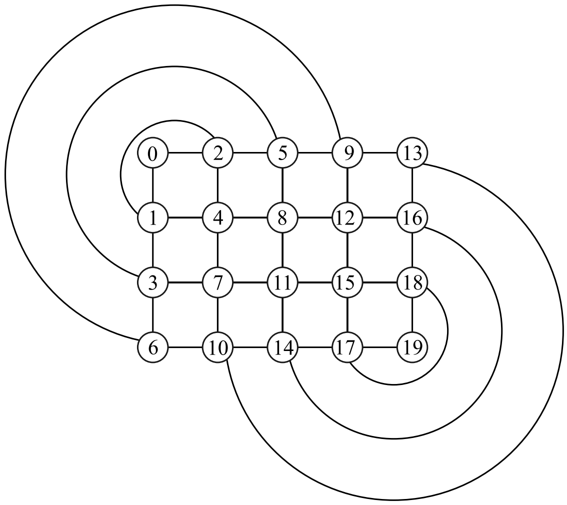

Fig. 2.7. A strongly indexable labeling $LJ_{4,5}$.

Definition 2.6. Let u be a vertex of $P_a \times P_b$ such that $deg(u) = 2$. Introduce an edge between every pair of distinct vertices v, w with $deg(v), deg(w) \neq 4$ if $d(u, v) = d(u, w)$ where $d(u, v)$ is the distance between u and v. The graph thus obtained is defined as the *level joined planar grid* and is denoted by $LJ_{a,b}$.

An example $LJ_{4,5}$ is illustrated in Fig. 2.7.

Theorem 2.7. *The graph $LJ_{a,b}$ is strongly indexable for all $a, b \geq 2$.*

Proof. Denote the vertex at ith row, jth column of $P_a \times P_b$ as $v_{i,j}$. Without loss of generality, we can assume $a \leq b$. Construct the graph $LJ_{a,b}$, as illustrated in Fig. 2.7. Let V be the vertex set of $LJ_{a,b}$ with p vertices.

Define a function $f : V \to \{0, 1, 2, \ldots, p-1\}$ such that

$$\begin{cases} f(v_{1,j}) = \dfrac{j(j+1)-2}{2}; & 1 \leq j \leq a \\ f(v_{1,j}) = \dfrac{(a+2aj-a^2)-2}{2}; & a < j \leq b \\ f(v_{i,j}) = f(v_{i-1,j+1}) - 1; & 2 \leq i \leq a,\ 1 \leq j \leq b+1-i \\ f(v_{i,j}) = f(v_{i-1,j}) + (a+b+1-j-i); & 2 \leq i \leq a,\ b+1-i < j \leq b. \end{cases}$$

One can verify that f thus defined is bijective. Also, one can observe that f is a *strongly indexable* labeling of $LJ_{a,b}$ for all $a, b \geq 2$. $\diamond$

For $a = 1, b = 2$ we get $P_1 \times P_2 \equiv P_2$ which is strongly indexable. A *strongly indexable* labeling of level joined planar grid $LJ_{4,5}$ using Theorem 2.7 is exhibited in Fig. 2.7.

Theorem 2.7 motivates us to construct a higher order strongly indexable graph. Let $v^x_{i,j}$ denote the vertex $v_{i,j}$ in the xth copy of $LJ_{a,b}$. For any integer $t > 1$, construct a graph by joining the vertex $v^x_{a-1,b}$ to the vertices $v^{x+1}_{1,1}, v^{x+1}_{1,2}, v^{x+1}_{2,1}$; $1 \leq x < t$ and denote the resulting graph as $LJ^t_{a,b}$ (See Fig. 2.8.) The following result is the general form of Theorem 2.7.

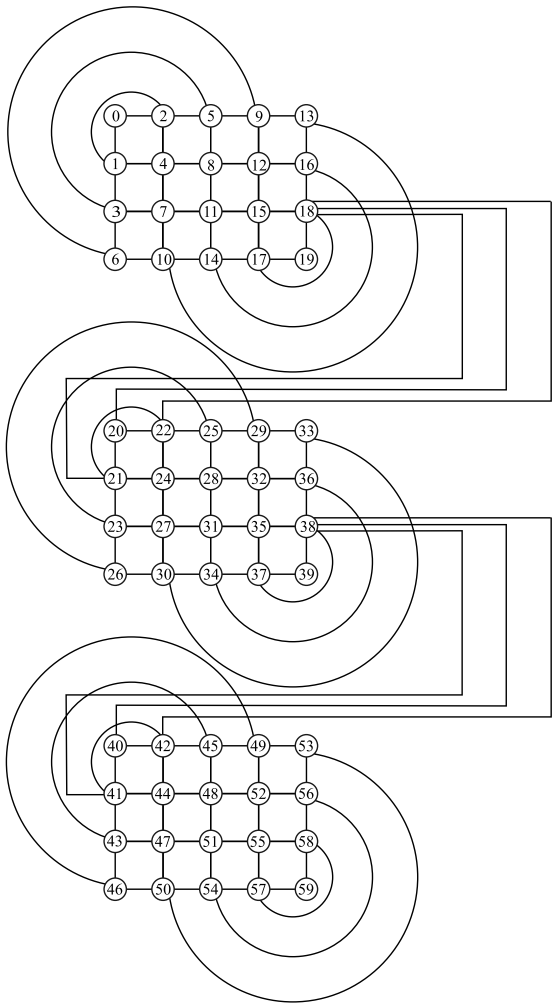

Fig. 2.8. A strongly indexable labeling of $LJ^3_{4,5}$.

Theorem 2.8. *The graph $LJ^t_{a,b}$ is strongly indexable for all integers $a, b \geq 2$ and $t \geq 1$.*

Proof. Denote the vertex of the graph $LJ^t_{a,b}$ as illustrated. One can observe that number of vertices of $LJ^t_{a,b}$ is equal to abt and the number of edges is equal to $2abt - 3$. That is, $p = abt$ and $q = 2abt - 3$.

Define a function $f : V(LJ^t_{a,b}) \rightarrow \{0, 1, 3, \ldots, abt - 1\}$ for $1 \leq x \leq t$, by

$$\begin{cases} f(v^x_{1,j}) = \dfrac{j(j+1)-2}{2} + ab(x-1); & 1 \leq j \leq a \\ f(v^x_{1,j}) = \dfrac{(a+2aj-a^2)-2}{2} + ab(x-1); & a < j \leq b \\ f(v^x_{i,j}) = f(v^x_{i-1,j+1}) - 1; & 2 \leq i \leq a,\ 1 \leq j \leq b+1-i \\ f(v^x_{i,j}) = f(v^x_{i-1,j}) + (a+b+1-j-i); & 2 \leq i \leq a,\ b+1-i < j \leq b. \end{cases}$$

Then one can verify that f thus defined is a strongly indexable labeling of the graph $LJ^t_{a,b}$ for all $a, b \geq 2$ and $t \geq 1$. Hence the graph $LJ^t_{a,b}$ is strongly indexable for all integers $a, b \geq 2$ and $t \geq 1$. ◇

For example, a strongly indexable labeling of $LJ^t_{a,b}$ when $a = 4$, $b = 5$, $t = 3$ using Theorem 2.8, is illustrated in Fig. 2.8.

3. Applications of strongly k-indexable graphs

In this section, we give a construction of a polygon having same internal angles and distinct sides using the strongly k-indexable labelings of a cycle.

Figueroa-Centenoa et al., [5] have introduced the concept of super edge-magic deficiency of graphs.

Definition 3.1. *Super edge-magic deficiency* of a graph G is the minimum number of isolated vertices added to G so that the resulting graph is super edge-magic and is denoted by $\mu_s(G)$.

Since a graph is super edge-magic if and only if it is strongly k-indexable for some k, super edge-magic deficiency is the minimum number of isolated vertices added to a graph G so that the resulting graph is strongly k-indexable for some k. One can observe that the results on super edge-magic deficiency of graphs can be proved in a simpler way using the concept of strongly k-indexable labeling. Hence, for the sake of convenience, we call this parameter *vertex dependent characteristic* and denote it by $d_c(G)$.

As there are many graphs which are not strongly k-indexable (see Acharya and Hegde [3,8]), it is interesting to study vertex dependent characteristics of graphs.

Figueroa-Centenoa et al., [5] have proved that

Theorem 3.2. *The vertex dependent characteristic of C_n is*

$$d_c(C_n) = \begin{cases} 1 & \text{if } n \equiv 0 (\text{mod } 4) \\ 0 & \text{if } n \text{ is odd} \\ \infty & \text{if } n \equiv 2 (\text{mod } 4). \end{cases}$$

Theorem 3.3. *The vertex dependent characteristic of the complete bipartite graph $K_{m,n}$ is at most $(m-1)(n-1)$.*

They conjectured that

Conjecture 3.4. *The vertex dependent characteristic of the complete bipartite graph $K_{m,n}$ is equal to $(m-1)(n-1)$.*

Theorem 3.3 can equivalently be stated as:

If the polynomial $f(x) = x^m + x^{m+1} + x^{m+2} + \cdots + x^{m+mn-1}$ (or $f(x) = x^n + x^{n+1} + x^{n+2} + \cdots + x^{n+mn-1}$) can be expressed as a product of two polynomials $p(x)$ and $q(x)$ having m and n terms respectively, such that no given power of x contained in both $p(x)$ and $q(x)$ then $d_c(K_{m,n}) \le (m-1)(n-1)$.

For example, consider $K_{4,6}$ and the polynomial $f(x) = x^6 + x^7 + x^8 + \cdots + x^{29}$. One can see that $f(x)$ can be expressed as a product of $p(x) = 1 + x + x^4 + x^5$ and $q(x) = x^6 + x^8 + x^{14} + x^{16} + x^{22} + x^{24}$. Assign the indices of x from $p(x)$ and $q(x)$ to the four and six vertices of $K_{4,6}$ respectively in a one-to-one manner. Then one can see that the numbers from 6 to 29 will appear on the edges of $K_{4,6}$. In all, there are 25 numbers from 0 to 24, and 10 numbers are assigned to the vertices. By introducing 15 isolated vertices and assigning the remaining 15 numbers in a one-to-one manner, one can see that $K_{4,6}$ is embedded as an induced subgraph of a strongly 6-indexable graph. This also shows that the vertex dependent characteristic of the complete bipartite graph $K_{4,6}$ is less than or equal to 15. We strongly believe that, using the above technique, Theorem 3.3 can be proved.

Construction of polygon $\mathcal{P}_{4n+2}$ with $4n+2$ sides such that all the internal angles are equal and lengths of the sides are distinct.

We know that C_{2n+1} is strongly k-indexable for $k = n$. Consider a regular polygon $\mathcal{P}_{2n+1}$ where the lengths of all sides are equal. Let f be a strongly k-indexable labeling of $C_{2n+1} \equiv \mathcal{P}_{2n+1}$. The construcion of $\mathcal{P}_{4n+2}$ with $4n+2$ sides such that all the internal angles are equal and lengths of the sides are distinct for all positive integer n using strongly k-indexable labeling of cycle graph C_{2n+1} is as follows.

1. *Step I:* Replace $f(u)$ by $f(u)+1$ for each $u \in V(\mathcal{P}_{2n+1})$ and then denote the vertex u of $\mathcal{P}_{2n+1}$ with label i as v_i.
2. *Step II:* Divide each side of $\mathcal{P}_{2n+1}$ into $4n+k+4$ equal parts.
3. *Step III:* Denote the point which is at i parts distance from the vertex v_i as v_{ij} if v_j is adjacent to v_i. Since v_i is adjacent to two vertices, we get the points v_{ij}, v_{ik} on the edges v_iv_j and v_iv_k. Join v_{ij} to v_{ik} so that $v_{ij}v_{ik}$ is an edge of $\mathcal{P}_{4n+2}$.
4. *Step IV:* Apply Step III to every vertex of $\mathcal{P}_{2n+1}$

From the above steps, we get $\mathcal{P}_{4n+2}$ and clearly each internal angle of $\mathcal{P}_{4n+2}$ so constructed is equal to $(\frac{2n}{2n+1})\pi$.

Also, lengths of the sides of $\mathcal{P}_{4n+2}$ are of the form

(i) $4n+k+4-(i+j)$ if the side of $\mathcal{P}_{4n+2}$ is the portion of the edge v_iv_j of C_{2n+1} or of the form

(ii) $\sqrt{i^2+i^2-2i^2\cos[(\frac{2n-1}{2n+1})\pi]}$ if the side of $\mathcal{P}_{4n+2}$ is the join of v_{ij} and v_{ik}. That is of the form it where $t = \sqrt{2(1-\cos[(\frac{2n-1}{2n+1})\pi])}$ and t is unique for every n.

From (i) and (ii) one can prove that lengths of the sides of $\mathcal{P}_{4n+2}$ are distinct.

Polygons $\mathcal{P}_6$ and $\mathcal{P}_{10}$ constructed from the strongly k-indexable graphs C_3 and C_5 using above method are displayed in Figs. 3.1 and 3.2.

From Theorem 3.2 we know that $C_{4n} \cup K_1$ is strongly k-indexable. Therefore the same method can be used in the construction of polygons $\mathcal{P}_{8n}$ from C_{4n} such that all the internal angles are the same and lengths of the sides are distinct. An example is displayed in Fig. 3.3.

Note that C_{4n+2} is not strongly k-indexable and $d_c(C_{4n+2}) = \infty$. Therefore, another natural question is "What is the minimum number of isolated edges that need to be added to a graph G so that the resulting graph is strongly k-indexable?".

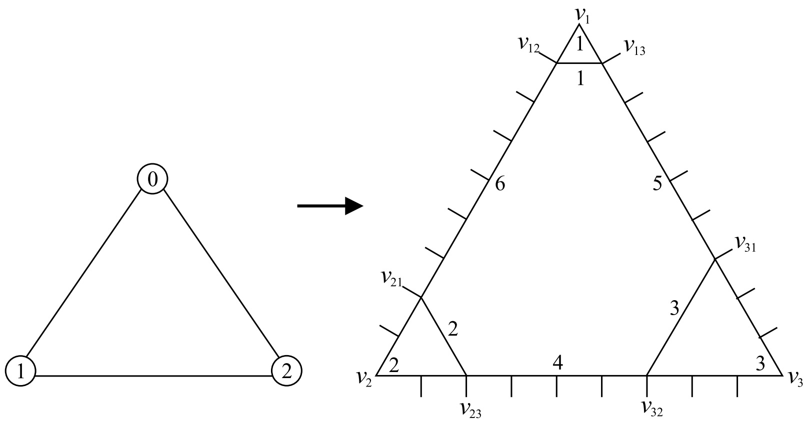

Fig. 3.1. Polygon $\mathscr{P}_6$ constructed form C_3.

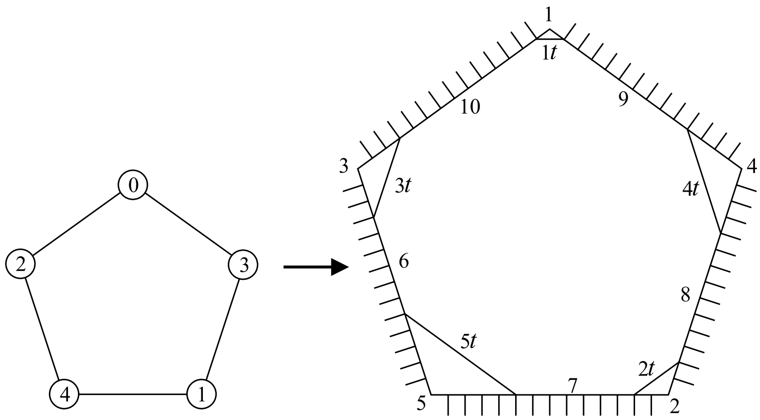

Fig. 3.2. Polygon $\mathscr{P}_{10}$ constructed form C_5 (note that $t = 1.618033989$).

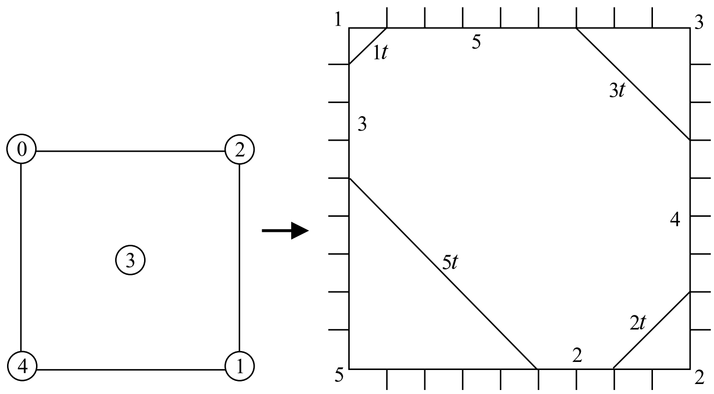

Fig. 3.3. Polygon $\mathscr{P}_8$ constructed form $C_4(t = \sqrt{2})$.

Definition 3.5. The minimum number of isolated edges added to a graph G so that the resulting disconnected graph which is strongly k-indexable is called the *edge dependent characteristic* and is denoted by $e_c(G)$. If a graph G is not strongly k-indexable by adding any number of isolated edges then $e_c(G) = \infty$ and if G is strongly k-indexable then $e_c(G) = 0$.

For example, $e_c(C_6) = 1$ (see Fig. 3.4) (Note that $d_c(C_6) = \infty$ by Theorem 3.2)
An example for polygon $\mathscr{P}_{12}$ obtained from C_6 is displayed in Fig. 3.5.

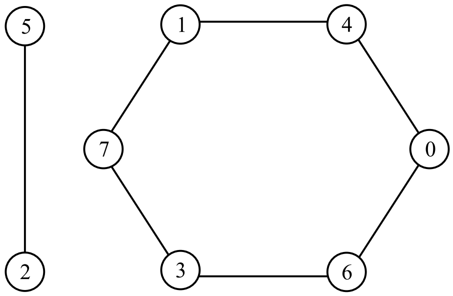

Fig. 3.4. A strongly 4-indexable $C_6 \cup 1K_2$.

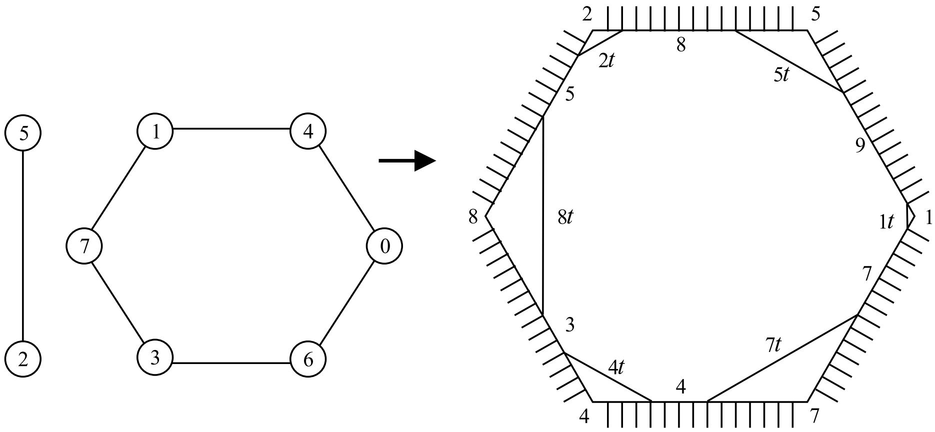

Fig. 3.5. Polygon $\mathscr{P}_{12}$ constructed from $C_6(t = \sqrt{3})$.

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