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Quotient Operators and the Open Mapping Theorem

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Abstract. Quotients of bounded operators on normed spaces have been discussed. Open mapping theorem for quotients of bounded operators and its consequences are given.

1. Introduction

Saichi Izumino introduced the notion of quotient of bounded operators on a Hilbert space and showed explicit formulae for computing the quotients which correspond to the sum, product, closure, adjoint and weak adjoint of given quotients [2–4]. A quotient is then possibly an unbounded linear operator and is what was called "semiclosed operator" by Kaufman [5]. In fact, the quotient of two compact operators need not be compact [1]. In this paper we explore quotients on normed spaces and find sufficient conditions for the set of quotients being complete. We prove open mapping theorem for quotients of bounded operators.

We conclude this introduction by establishing some notation and terminology. We consider normed spaces X, Y, Z over a field $\mathbb{K}$ of real or complex scalars. The set of linear operators from a normed space X to a normed space Y will be denoted by $\mathcal{L}(X, Y)$ and the set of bounded linear operators will be denoted by $\mathcal{B}(X, Y)$. If $T \in \mathcal{L}(X, Y)$, then $\mathcal{R}(T)$ and $\mathcal{N}(T)$ denote range and null space of T respectively.

2. Concepts and Results

Definition 2.1. *[2] Let X, Y, Z be normed spaces and $A \in \mathcal{L}(X, Y)$, $B \in \mathcal{L}(X, Z)$ with $\mathcal{N}(A) \subseteq \mathcal{N}(B)$. The mapping $Ax \mapsto Bx$ defined on $\mathcal{R}(A)$ is called a **quotient** of linear operators B and A and is denoted by B/A. The condition $\mathcal{N}(A) \subseteq \mathcal{N}(B)$ is described by saying that B is **determined by** A. Given $A \in \mathcal{B}(X, Y)$, $B \in \mathcal{B}(X, Z)$, a stronger version is that for some $k > 0$, $\|Bx\| \leq k\|Ax\|$, for every $x \in X$, which says that B is **majorized by** A.*

Let $A \in \mathcal{L}(X, Y)$ be fixed. We form a set, say $\mathcal{L}_A(X, Z)$ the set of all linear operators B from X to Z such that B/A is defined. That is,

$$\mathcal{L}_A(X, Z) = \left\{B \in \mathcal{L}(X, Z) : \mathcal{N}(A) \subseteq \mathcal{N}(B)\right\}.$$

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Note that $\mathcal{L}_A(X,Z)$ contains the operator A and the zero operator from X to Z. It is easy to show that $\mathcal{L}_A(X,Z)$ is a subspace of $\mathcal{L}(X,Z)$, for each fixed A in $\mathcal{L}(X,Y)$. Motivated by the space $\mathcal{L}_A(X,Z)$, for a fixed $A \in \mathcal{L}(X,Y)$, we define the following set of quotient operators

$$\mathcal{Q}_A(\mathcal{R}(A),Z) := \left\{B/A : \mathcal{N}(A) \subseteq \mathcal{N}(B)\right\}.$$

Addition and scalar multiplication are defined by

$$\begin{aligned} B/A + C/A &= (B+C)/A \\ \alpha(B/A) &= \alpha B/A \end{aligned}$$

for $B/A, C/A$ in $\mathcal{Q}_A(\mathcal{R}(A),Z)$ and scalars α in $\mathbb{K}$. It can be proved that $\mathcal{Q}_A(\mathcal{R}(A),Z)$ is a linear space by verifying the null space inclusions. Generally

$$\mathcal{L}_A(X,Z) \subseteq \left\{B \in \mathcal{L}(X,Z) : \|B(.)\| \leq k\|A(.)\| \text{ for some } k > 0\right\} \subseteq \mathcal{L}(X,Z).$$

The second inclusion may fail if we replace $\mathcal{L}(.)$ by bounded operators $\mathcal{B}(.)$: without boundedness, we have

$$\mathcal{L}(\mathcal{R}(A),Z) = \mathcal{L}(Y,Z)|_{\mathcal{R}(A)} = \left\{B|_{\mathcal{R}(A)} : B \in \mathcal{L}(Y,Z)\right\}.$$

For Banach spaces, if $\mathcal{R}(A)$ is not closed then it cannot be another Banach space, although it is an "operator range."

Theorem 2.2. *Let X, Y, Z be linear spaces and let $A : X \to Y$ be linear. If X and Z are of finite dimensions, then $dim(\mathcal{Q}_A(\mathcal{R}(A),Z)) = dim(\mathcal{R}(A))\, dim(Z)$.*

Proof. The relation holds good when A is the zero operator. Let $dim(X) = n$, $dim(Z) = m$ and $dim(\mathcal{N}(A)) = k$. Then $dim(\mathcal{R}(A)) = n - k$. By rank-nullity theorem, we observe that the quantity in the right side of the last equality is finite. When A is injective, B/A can be formed for every member of $\mathcal{L}(X,Z)$. Thus $nm = dim(\mathcal{Q}_A(\mathcal{R}(A),Z)) = dim(\mathcal{L}(X,Z)) = nm$.

We now assume that $\mathcal{N}(A)$ and $\mathcal{R}(A)$ are nonzero subspaces. Let $\{v_1, v_2, \ldots, v_k\}$ be a basis of $\mathcal{N}(A)$. We extend this to get a basis of X, say $\{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$. Let $\{w_1, w_2, \ldots, w_m\}$ be a basis of Z. For $1 \leq p \leq m$, $1 \leq q \leq n-k$, define $T_{p,q} : X \to Z$ by

$$T_{p,q}(v_i) = \delta_{k+q,i}\, w_p$$

for $i = 1, \ldots, n$. As $\mathcal{N}(A) \subseteq \mathcal{N}(T_{p,q})$ for $1 \leq p \leq m$, $1 \leq q \leq n-k$, $T_{p,q}/A \in \mathcal{Q}_A(\mathcal{R}(A),Z)$. Let $B/A \in \mathcal{Q}_A(\mathcal{R}(A),Z)$. Then $B(v_i) = 0, i = 1, \ldots, k$. Now for each $j \in \{k+1, k+2, \ldots, n\}$

$$Bv_j = \sum_{\ell=1}^{m} \alpha_{\ell,j-k}\, w_\ell$$

for some scalars $\alpha_{p,q}$, $1 \leq p \leq m$, $1 \leq q \leq n-k$. To prove $B/A = \sum_{p=1}^{m}\sum_{q=1}^{n-k} \alpha_{p,q}(T_{p,q}/A)$ on $\mathcal{R}(A)$, it is enough to prove the relation at each basis element of $\mathcal{R}(A)$. Let $\{u_1, \ldots, u_{n-k}\}$ be a basis of $\mathcal{R}(A)$. For each fixed j in $1 \leq j \leq n-k$, there is some $x \in X$ such that $u_j = Ax$. Let $x = \beta_1 v_1 + \cdots + \beta_n v_n$, for unique scalars $\beta_1, \ldots, \beta_n$. Then $u_j = \beta_{k+1} A v_{k+1} + \cdots + \beta_n A v_n$. Consider

$$\begin{aligned} (B/A)u_j = Bx &= \beta_{k+1} B v_{k+1} + \cdots + \beta_n B v_n \\ &= \beta_{k+1}(\alpha_{1,1} w_1 + \cdots + \alpha_{m,1} w_m) + \cdots + \beta_n(\alpha_{1,n-k} w_1 + \cdots + \alpha_{m,n-k} w_m) \end{aligned}$$

and

$$\begin{aligned}\sum_{p=1}^{m}\sum_{q=1}^{n-k}\alpha_{p,q}(T_{p,q}/A)u_j &= \sum_{p=1}^{m}\sum_{q=1}^{n-k}\alpha_{p,q}T_{p,q}(x)\\ &= \sum_{p=1}^{m}\sum_{q=1}^{n-k}\alpha_{p,q}(\beta_{k+1}T_{p,q}(v_{k+1})+\cdots+\beta_n T_{p,q}(v_n))\\ &= \beta_{k+1}(\alpha_{1,1}w_1+\cdots+\alpha_{m,1}w_m)+\cdots+\beta_n(\alpha_{1,n-k}w_1+\cdots+\alpha_{m,n-k}w_m).\end{aligned}$$

Therefore $\left\{T_{p,q}/A : 1\le p\le m, 1\le q\le n-k\right\}$ spans $\mathcal{Q}_A(\mathcal{R}(A),Z)$.

Suppose

$$\sum_{i=1}^{m}\alpha_{i,1}(T_{i,1}/A)+\cdots+\alpha_{i,n-k}(T_{i,n-k}/A)=0/A$$

for some scalars $\alpha_{i,j}, 1\le i\le m,\ 1\le j\le n-k$. Then

$$\sum_{i=1}^{m}(\alpha_{i,1}T_{i,1}+\cdots+\alpha_{i,n-k}T_{i,n-k})/A=0/A.$$

Evaluating $A(v_j)$, $k+1\le j\le n$, in the above operator relation, we get $\sum_{i=1}^{m}\alpha_{i,1}w_i=0$. Hence $\alpha_{i,j}=0$, for all $1\le i\le m,\ 1\le j\le n-k$. □

Theorem 2.3. *Let X be a division Banach algebra (with identity) over the complex plane $\mathbb{C}$. Then*

$$dim(\mathcal{Q}_A(\mathcal{R}(A),Z)) = dim(X)\,dim(Z) \text{ or } (dim(X)-1)\,dim(Z).$$

Proof. Gelfand mapping theorem [6] says that every complex Banach division algebra is isometrically isomorphic to the complex plane $\mathbb{C}$. As the null space of the non-zero operator A is of dimension zero or one, the proof follows. □

We now discuss a collection of bounded operators for a fixed linear operator A from X to Z. We define

$$\mathcal{B}_A(X,Z)=\left\{B\in\mathcal{B}(X,Z):\mathcal{N}(A)\subseteq\mathcal{N}(B)\right\}.$$

Presence of the zero operator in $\mathcal{B}_A(X,Z)$ proves that $\mathcal{B}_A(X,Z)$ is non-empty. It is easy to verify that for normed spaces X,Z, $\mathcal{B}_A(X,Z)$ is a closed subspace of $\mathcal{B}(X,Z)$. The following result tells that if X,Z are also Banach spaces, then so is $\mathcal{B}_A(X,Z)$.

Theorem 2.4. *Let X,Z be Banach spaces. The set $\mathcal{B}_A(X,Z)$ of bounded linear operators B with $\mathcal{N}(A)\subseteq\mathcal{N}(B)$ is a Banach space.*

Proof. Let $\{B_n\}$ be a Cauchy sequence in $\mathcal{B}_A(X,Z)$. Since $\mathcal{B}(X,Z)$ is complete, there exists $B\in\mathcal{B}(X,Z)$ such that $B_n\to B$ in the norm of $\mathcal{B}(X,Z)$. Let $x\in\mathcal{N}(A)$. Since for each n, $B_n\in\mathcal{B}_A(X,Z)$, we have $B_n(x)=0$. Convergence of the sequence $\{B_nx-Bx\}$ to zero in Z shows that $x\in\mathcal{N}(B)$. □

Under the above settings, let us take $\mathcal{B}_A^b(X,Z)=\left\{B\in\mathcal{B}_A(X,Z): B/A \text{ is bounded}\right\}$. Since $0/A$ is bounded, the set given above is nonempty.

Theorem 2.5. *$\mathcal{B}_A^b(X,Z)$ is a normed subspace of $\mathcal{B}_A(X,Z)$.*

Proof. First it is required to prove that $\mathcal{B}_A^b(X,Z)$ is closed under linear space operations.
Let α be a scalar and $B, C \in \mathcal{B}_A^b(X,Z)$. Note that for any $D \in \mathcal{B}_A^b(X,Z)$

$$\begin{aligned} \|D/A\| &= \sup\left\{\|(D/A)y\| : y \in \mathcal{R}(A), \|y\| \le 1\right\} \\ &= \sup\left\{\|Dx\| : x \in X, \|Ax\| \le 1\right\}. \end{aligned}$$

Let $x \in X$ be such that $\|Ax\| \le 1$. Therefore

$$\begin{aligned} \|(B+C)x\| &\le \|Bx\| + \|Cx\| \le \|B/A\| + \|C/A\| \quad \text{and} \\ \|(\alpha B)x\| &= |\alpha|\,\|Bx\| \le |\alpha|\,\|B/A\| \end{aligned}$$

imply that

$$\begin{aligned} \|(B+C)/A\| &\le \|B/A\| + \|C/A\| < \infty \quad \text{and} \\ |\alpha(B/A)\| &\le |\alpha|\,\|B/A\| < \infty. \end{aligned}$$

□

Theorem 2.6. *Let Z be a Banach space. If the set $\left\{x \in X : \|Ax\| \le 1\right\}$ is bounded in X, then $\mathcal{B}_A^b(X,Z)$ is Banach.*

Proof. Let $\{B_n\}$ be a Cauchy sequence in $\mathcal{B}_A^b(X,Z)$. Since $\mathcal{B}_A(X,Z)$ is complete, there exists $B \in \mathcal{B}_A(X,Z)$ such that $B_n \to B$ in the norm of $\mathcal{B}_A(X,Z)$. Now it suffices to show that B/A is bounded. Let $M = \sup\{\|x\| : x \in X, \|Ax\| \le 1\}$ and $x \in X$ be such that $\|Ax\| \le 1$. Since $\{\|B_n\|\}$ converges, there exists K such that $\|B_n\| \le K$ for all n. Choose N such that $\|(B_n - B)x\| \le 1$ for all $n \ge N$. Then for each $x \in X$ such that $\|Ax\| \le 1$,

$$\|Bx\| \le \|B_n x\| + 1 \le \|B_n\|\,\|x\| + 1 \le KM + 1,$$

for all $n \ge N$. Therefore $\|B/A\| = \sup\left\{\|Bx\| : x \in X, \|Ax\| \le 1\right\}$ is finite. □

Let $A \in \mathcal{B}(X,Y)$ be fixed. Now let us take $\mathcal{Q}_A^b(\mathcal{R}(A),Z) = \left\{B/A \in \mathcal{Q}_A(\mathcal{R}(A),Z) : B, B/A \text{ are bounded}\right\}$. We now show that $\mathcal{Q}_A^b(\mathcal{R}(A),Z)$ is Banach when Z is a Banach space and $\left\{x \in X : \|Ax\| \le 1\right\}$ is a bounded set in X.

Theorem 2.7. *$\mathcal{Q}_A^b(\mathcal{R}(A),Z)$ is a normed subspace of $\mathcal{B}_A(X,Z)$.*

Proof. We start first by showing that the operator norm is a norm on $\mathcal{Q}_A^b(\mathcal{R}(A),Z)$.
Let $B/A \in \mathcal{Q}_A^b(\mathcal{R}(A),Z)$ be such that $\|B/A\| = 0$. Then $Bx = 0$ for all x in X such that $Ax \ne 0$. Whenever $Ax = 0$, due to the nullspace inclusion, we have $Bx = 0$. Thus B is the zero operator. Hence $B/A = 0/A$.

Let $B/A \in \mathcal{Q}_A^b(\mathcal{R}(A),Z)$ and α be a scalar. Now

$$\begin{aligned} \|\alpha B/A\| &= \sup\left\{\|(\alpha B)x\| : x \in X, \|Ax\| \le 1\right\} \\ &= |\alpha| \sup\left\{\|Bx\| : x \in X, \|Ax\| \le 1\right\} = |\alpha|\,\|B/A\|. \end{aligned}$$

Let $B/A, C/A \in \mathcal{Q}_A^b(\mathcal{R}(A),Z)$. Then

$$\begin{aligned} \|B/A + C/A\| &= \sup\left\{\|Bx + Cx\| : x \in X, \|Ax\| \le 1\right\} \\ &\le \sup\left\{\|Bx\| : x \in X, \|Ax\| \le 1\right\} + \sup\left\{\|Cx\| : x \in X, \|Ax\| \le 1\right\} \\ &= \|B/A\| + \|C/A\|. \end{aligned}$$

□

Theorem 2.8. *Let Z be a Banach space. If $\left\{x \in X : \|Ax\| \le 1\right\}$ is a bounded set in X, then $\mathcal{Q}_A^b(\mathcal{R}(A),Z)$ is Banach.*

Proof. Let $\{B_n/A\}$ be a Cauchy sequence in $\mathcal{Q}_A^b(\mathcal{R}(A), Z)$. We claim that $\{B_n\}$ is a Cauchy sequence in $\mathcal{B}(X, Z)$. Let $\varepsilon > 0$ be given. Since $\{B_n/A\}$ is Cauchy, there exists N such that

$$\|B_n/A - B_m/A\| < \frac{\varepsilon}{\|A\| + 1} \quad \text{for all } n, m \geq N.$$

That is, for all $n, m > N$,

$$\sup\left\{\|(B_n - B_m)x\| : x \in X, \|Ax\| \leq 1\right\} < \frac{\varepsilon}{\|A\| + 1}.$$

Let x be an element in X such that $\|x\| \leq 1$. Then $\|Ax\| \leq \|A\|$. Now for all $n, m > N$,

$$\begin{aligned}
\|(B_n - B_m)x\| &= \|((B_n - B_m)/A)Ax\| \\
&\leq \|(B_n - B_m)/A\| \, \|Ax\| \\
&\leq \|(B_n - B_m)/A\| \, \|A\| \\
&\leq \frac{\varepsilon}{\|A\| + 1}\|A\| < \varepsilon.
\end{aligned}$$

Thus $\|B_n - B_m\| = \sup\{\|(B_n - B_m)x\| : x \in X, \|x\| \leq 1\} < \varepsilon$ for all $n, m > N$. Let B be the linear uniform limit of $\{B_n\}$ and $M = \sup\{\|x\| : x \in X, \|Ax\| \leq 1\}$. By Theorem 2.6, we see that B/A is bounded. For a given $\varepsilon > 0$, convergence of $\{B_n\}$ to B gives an integer N such that $\|B - B\| < \frac{\varepsilon}{M+1}$ for all $n > N$. Thus for all $n > N$,

$$\begin{aligned}
\|B_n/A - B/A\| &= \|(B_n - B)/A\| \\
&= \sup\left\{\|(B_n - B)x\| : x \in X, \|Ax\| \leq 1\right\} \\
&\leq \sup\left\{\|B_n - B\| \, \|x\| : x \in X, \|Ax\| \leq 1\right\} \\
&= \|B_n - B\| \, \sup\left\{\|x\| \in X : \|Ax\| \leq 1\right\} \\
&= \|B_n - B\|M < \varepsilon.
\end{aligned}$$

□

Example 2.9. *The condition* $\left\{x \in X : \|Ax\| \leq 1\right\}$ *is bounded in X is sufficient but it is not a necessary condition. Consider the zero map on* $\mathbb{R}$*. Now* $\mathcal{Q}_0^b(\{0\}, \mathbb{R})$ *being singleton is a Banach space but the set* $\left\{x \in \mathbb{R} : |0x| \leq 1\right\} = \mathbb{R}$ *is not bounded with the usual metric.*

We denote $\mathcal{Q}_A^b(\mathcal{R}(A), Z)$ by $\mathcal{Q}_A^b(\mathcal{R}(A))$ when $X = Y = Z$. $\mathcal{Q}_A(\mathcal{R}(A))$ is an algebra with respect to the multiplication defined by $(B/A).(C/A) = (BC)/A$.

Theorem 2.10. *If* $\left\{x \in X : \|Ax\| \leq 1\right\}$ *is bounded in a Banach space X and* $\|A\| \leq 1$*, then* $\mathcal{Q}_A(\mathcal{R}(A))$ *is a Banach algebra.*

Proof. Let B/A, $C/A \in \mathcal{Q}_A(\mathcal{R}(A))$. Now

$$\begin{aligned}
\|(B/A).(C/A)\| &= \sup\left\{\|(BC)x\| : x \in X, \|Ax\| \leq 1\right\} \\
&= \sup\left\{\|(B/A)ACx\| : x \in X, \|Ax\| \leq 1\right\} \\
&\leq \sup\left\{\|B/A\| \, \|(AC)x\| : x \in X, \|Ax\| \leq 1\right\} \\
&= \|B/A\| \sup\left\{\|(AC)x\| : x \in X, \|Ax\| \leq 1\right\} \\
&\leq \|B/A\| \sup\left\{\|A\| \, \|Cx\| : x \in X, \|Ax\| \leq 1\right\} \\
&= \|B/A\| \, \|A\| \sup\left\{\|Cx\| : x \in X, \|Ax\| \leq 1\right\} \\
&= \|B/A\| \, \|A\| \, \|C/A\| \leq \|B/A\| \, \|C/A\|.
\end{aligned}$$

□

3. Open Mapping Theorem and Its Consequences

We now prove the open mapping theorem for quotient operators without using its boundedness. We derive the following lemma first.

Lemma 3.1. *Let X, Y, Z be Banach spaces and let $A \in \mathcal{B}(X, Y)$ with closed range, and $B \in \mathcal{B}(X, Z)$ with $\mathcal{N}(A) \subseteq \mathcal{N}(B)$. If $\overline{(B/A)(B_r(0))}$ contains an open ball centered at the origin in Z, then*

$$\overline{(B/A)(B_r(0))} \subseteq (B/A)(B_{2r}(0)).$$

Proof. Suppose $B_s(0) \subseteq \overline{(B/A)(B_r(0))}$ for some $s > 0$. Let $w \in \overline{(B/A)(B_r(0))}$. We choose a sequence $\{v_n\}$ in $B_r(0)$ such that

$$\Big\|w - \sum_{i=1}^{n} \frac{1}{2^{i-1}}(B/A)(v_i)\Big\| = \Big\|w - (B/A)\Big(\sum_{i=1}^{n} \frac{1}{2^{i-1}} v_i\Big)\Big\| < \frac{s}{2^n}$$

for all n. Since $\left\{ \sum_{i=1}^{n} \frac{1}{2^{i-1}} v_i \right\}$ is Cauchy and $\mathcal{R}(A)$ is complete, $\sum_{i=1}^{\infty} \frac{1}{2^{i-1}} v_i \in \mathcal{R}(A)$. Each v_i in $B_r(0)$ gives $\| \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} v_i \| < 2r$. For each i, let $u_i = \frac{1}{2^{i-1}} A^{-1}(v_i)$. The open mapping theorem produces A as a homeomorphism from X to $\mathcal{R}(A)$.

Now

$$\begin{aligned} \|u_1 + \cdots + u_n\| &= \Big\|A^{-1}\Big(\sum_{i=1}^{n} \frac{1}{2^{i-1}} v_i\Big)\Big\| \\ &\leq \|A^{-1}\| \; \Big\|\sum_{i=1}^{n} \frac{1}{2^{i-1}} v_i\Big\| \end{aligned}$$

implies $\left\{ \sum_{i=1}^{n} u_i \right\}$ is Cauchy; and hence $\left\{ \sum_{i=1}^{\infty} u_i \right\} \in X$. Continuity of A^{-1} and B give $\sum_{i=1}^{\infty} u_i = A^{-1}(\sum_{i=1}^{\infty} \frac{1}{2^{i-1}} v_i)$ and $B(\sum_{i=1}^{\infty} u_i) = \sum_{i=1}^{\infty} B(u_i)$ respectively. Thus

$$\begin{aligned} w &= \sum_{i=1}^{\infty} (B/A)\Big(\frac{1}{2^{i-1}} v_i\Big) = \sum_{i=1}^{\infty} (B/A)\Big(\frac{1}{2^{i-1}} v_i\Big) \\ &= \sum_{i=1}^{\infty} (B/A)A(u_i) = \sum_{i=1}^{\infty} B(u_i) \\ &= B\Big(\sum_{i=1}^{\infty} u_i\Big) = (B/A)\Big(A\Big(\sum_{i=1}^{\infty} u_i\Big)\Big) \\ &= (B/A)\Big(\sum_{i=1}^{\infty} \frac{1}{2^{i-1}} v_i\Big). \end{aligned}$$

□

Theorem 3.2. *Let X, Y, Z be Banach spaces. Let $A \in \mathcal{B}(X, Y)$ be injective and $B \in \mathcal{B}(X, Z)$ be surjective. If $\mathcal{N}(A) \subseteq \mathcal{N}(B)$ and $\mathcal{R}(A)$ is closed, then B/A is an open mapping on $\mathcal{R}(A)$.*

Proof. Let G be a nonempty open set in $\mathcal{R}(A)$ and $w \in (B/A)(G)$. Now $w = (B/A)(v)$ for some $v \in G$. From the openness of G in $\mathcal{R}(A)$ there is some $r > 0$ such that $B_r(v) \subseteq G$; from the linearity of B/A, $(B/A)(v) + (B/A)(B_r(0)) = (B/A)(B_r(v)) \subseteq (B/A)(G)$.

Now, $(B/A)(B_r(0))$ contains an open ball centered at the origin. Surjectiveness of B pressures the same for (B/A) from range of A. Thus $Z = B(X) = (B/A)(\mathcal{R}(A)) = (B/A)(\cup_{n=1}^{\infty} B_n(0)) = \cup_{n=1}^{\infty} (B/A)(B_n(0))$. From Baire's theorem, we get $a > 0$ such that $B'_a(0) \subseteq (B/A)(B_1(0))$ and hence $B'_{ar}(0) \subseteq r(B/A)(B_1(0)) = (B/A)(B_r(0))$. By Lemma 3.1, $\overline{(B/A)(B_{n_0}(0))} \subseteq (B/A)(B_{2n_0}(0))$. So $(B/A)(G) \supset (B/A)(v) + B'_{ar}(0) = B'_{ar}(w)$. □

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