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OPTIMUM SHAPE DESIGN OF ROTATING DISKS

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Abstract–This paper deals with optimum shape design of the rotating disks by nonlinear programming method. The shape of the cross section is defined by 5th degree polynomial which is completely determined by the boundary conditions and four design variables. The stress analysis of the disk is carried out by finite element method using isoparametric elements. The optimization technique used is with improved movelimit method of sequential linear programming. Progress of optimization is investigated with three different objective functions. After preliminary studies a weighted objective function is selected for detailed investigation. Optimum shapes are obtained for different speeds and for different fit pressures from hub. The results are presented in non-dimensionalised form.

INTRODUCTION

High speed rotating disks are very commonly used as flywheels, gears, rotors in turbines and compressors. In the design of these rotating disks, the magnitude and distribution of the stresses constitute a major constraining factor. This problem has attracted the attention of many investigators[1–7]. For rotating disks uniform cross section is very uneconomical. Stodola[1] gave a comprehensive analysis of the problem and suggested a hyperobolic curve for the profile of the cross section of the disk. Donath[2] developed an approximate method which replaces the actual disk by a series of rings with uniform thickness. Grammel[3, 4] gave the further development of this method.

Mathematical programming approach to optimum design of rotating disks may be found in Refs. [5–7]. de Silve[5, 6] has optimized the cross section of rotating disk using constraints on frequency and stresses. Sometimes the disks with inner diameter of the hub smaller than the diameter of shaft are taken, heated and then fitted on to shaft. After cooling the shaft and the disks are fitted tightly and it gives rise to pressure between them. Ali Seireg and Surana[7] have considered this fit pressure in developing optimal designs. However in these investigations, the methods of stress analysis are not very accurate particularly in the region of stress concentration. The disk is approximated by a number of rings, the optimum thicknesses of which are determined by mathematical programming. The optimum profile is obtained by smoothening the stepped shape obtained. In the present investigation the curved cross section is defined by two symmetric polynomial curves which depend upon thicknesses at four predefined points. The semi thicknesses at these four predefined points are considered as design variables. A suitable subroutine is developed to generate automatically a quadratic isoparametric finite elements mesh for prescribed design variables. The stress analysis is carried out with these finite elements which are ideally suited for continuum involving curved boundaries. The optimization is carried out using a powerful automated optimum structural design program developed by the authors[8, 9]. The progress of optimization is studied with three different objective functions. Parametric studies of optimum design have been carried out for different speed and fit pressures.

GEOMETRIC DESCRIPTION OF CROSS SECTION OF ROTATING DISKS

Figure 1 shows a typical rotating disk with hub and rim. The following notations are used in the text: r_1 = Inner radius of hub; r_2 = Outer radius of hub; r_3 = Inner radius of rim; r_4 = Outer radius of rim; t_1 = Semi-width of hub; and t_2 = Semi width of rim.

Engineering design considerations specify the width of hub ($2t_1$) and the dimensions of rim. The outer radius of hub r_2 can be a variable, but however this is considered as a fixed parameter in this paper. The disk can have varying thickness between hub and rim. The present investigation is concerned with the design of optimum shape of this portion of rotating disk. The disk cross section is symmetric about the section A–A. The curved profile is expressed in the form of a polynomial in the variables x and y. The axes $0x$ and $0y$ with respect to which the coordinates x and y are measured are shown in Fig. 1. Thus

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 \quad (1)$$

in which $a_0, \ldots\ldots\ldots, a_5$ are arbitrary constants. The end conditions at $x = 0$ and $x = l$ (where $l = r_3 - r_2$) give

$$a_0 = 0 \quad (2)$$

and

$$t_1 - t_2 = a_1l + a_2l^2 + a_3l^3 + a_4l^4 + a_5l^5. \quad (3)$$

Four more conditions are required to determine all the

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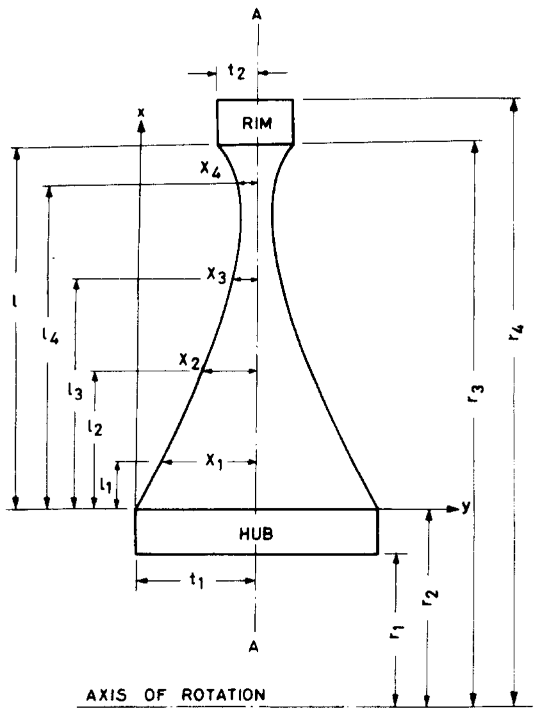

Fig. 1. A typical rotating disk.

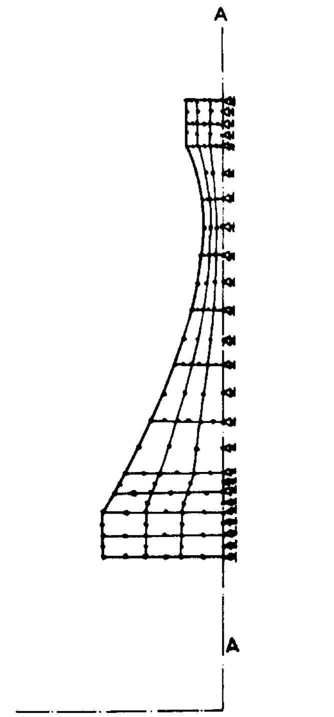

Fig. 2. Finite element idealization.

arbitrary constants. The semi-thicknesses at four specified points on the disk are selected as design variables. In Fig. 1, variables X_1, X_2, X_3 and X_4 are shown as the semithicknesses of disk at distances l_1, l_2, l_3, and l_4. Thus

$$t_1 - X_i = a_1 l_1 + a_2 l^2_i + a_3 l^3_i + a_4 l^4_i + a_5 l^5_i \qquad i = 1, 2, 3, 4. \quad (4)$$

From eqns (2)–(4) all arbitary constants are determined which define the shape of the curve completely. The shapes so obtained for various combinations of design variables are found to be reasonably good.

FINITE ELEMENT ANALYSIS

The stress analysis of rotating disk is carried out by finite element analysis using linearly elastic approach with quadratic isoparametric elements. The loading considered are the inertia force due to rotation and fit pressure between hub and the shaft. The loads on the rim are not considered. Hence it is an axi-symmetric problem and also it is sufficient to consider only half the cross section on account of the symmetry of loading and the geometry about axis A–A (Fig. 1). Figure 2, shows an automated mesh generated for finite element analysis. Thirty-six elements have been used to describe the continuum symmetry. A finer mesh is used near the hub since the stress concentration in this region is high. The symmetry boundary conditions are imposed on the nodal points along the line of symmetry A–A. The 2×2 Gaussian integration is employed for the evaluation of the integrals involved in finite element method. The finite element analysis is tested by analysing rotating disks of constant thicknesses and stodola shape and by comparison of results with solutions available in literature[4, 7]. The results are very satisfactory. The finite element method is capable of considering the variation of stresses through thickness. The tangential stresses are higher at the level of middle plane than at the fibres. Figure 3 shows stress distribution along section A–A in the central portion for a typical rotating disk. It may be observed that the tangential stress σ_θ is more prominent and is quite high near the hub. The higher value of radial stress σ_r occurs near rim.

MATHEMATICAL FORMULATION

A general constrained optimization problem may be stated as

$$\text{Min } z = F(X) \qquad (5)$$

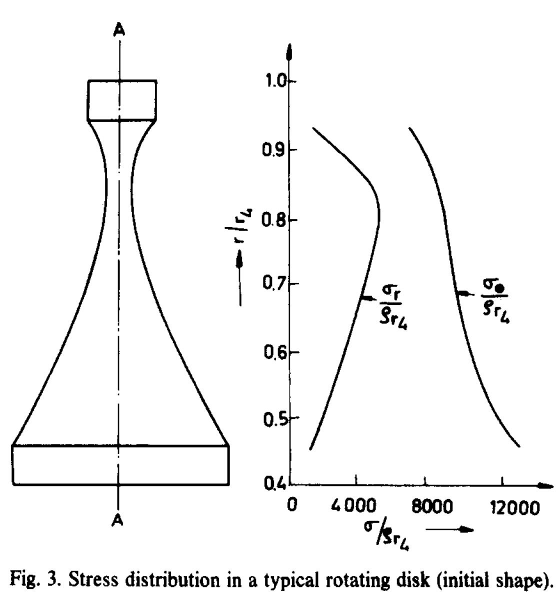

Fig. 3. Stress distribution in a typical rotating disk (initial shape).

$$\text{Subject to } G_j(X) \leq 0 \qquad j = 1, 2, \ldots\ldots m \tag{6}$$

where X is a design vector of n variables, $F(X)$ is objective function and $G_j s$ are constraints.

In the present investigation, as explained earlier, the semi-thickness at four pre-defined points are taken as the design variables. The four pre-defined points are at distances $0.125l$, $0.375l$, $0.625l$ and $0.875l$ from the origin 0.

In the design of rotating disk the best objective function is not well defined. De Silve[5,6] has taken minimization of weight as objective function. Ali Seirag and Surana[7] have carried out investigation with six objective functions and finally selected minimization of difference between maximum and minimum tangential stress as objective function for detailed study. In this investigation the following three objective functions are considered: (1) Minimization of difference between maximum and minimum tangential stress. (2) Stress levelling. (3) A weighted objective function having equal weight–age for volume minimization and stress leveling.

For numerical calculation of the above objective functions stresses at 16 sampling points (Fig. 4) are considered. The stresses at these sampling points are obtained by linearly extrapolating stresses at Gauss points in the elements. The calculation of various objective functions used in this investigation is explained below:

(1) *Minimization of difference between maximum and minimum tangential stresses*

The tangential stresses at 16 sampling points are scanned through to find maximum and minimum tangential stresses. Then the objective function is

$$F(X) = \sigma_{\theta\max} - \sigma_{\theta\min}. \tag{7}$$

The constraints are imposed that at any point

$$\sigma_r \geq \sigma_{\theta\max} \tag{8}$$

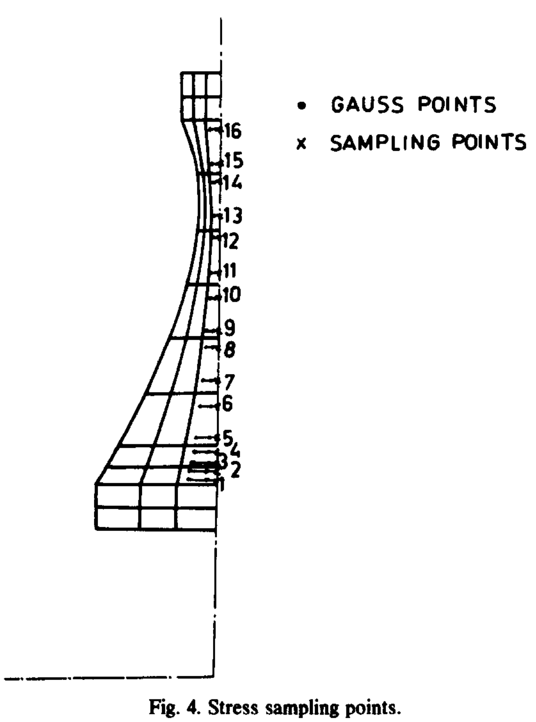

Fig. 4. Stress sampling points.

(2) *Stress levelling*

In this the objective is to get uniform stress along Section A–A. This objective is mathematically expressed as

$$F(X) = \oint (\sigma - \sigma_a)^2 \, ds \tag{9}$$

where σ is the maximum principal stress and σ_a is the average stress at initial shape. To non-dimensionalise this value and also to keep it within reasonable range it is modified as

$$F(X) = \frac{\oint(\sigma - \sigma_a)^2 \, ds}{\oint(\sigma_0 - \sigma_a)^2 \, ds} \tag{10}$$

where σ_0 correspond to stresses at initial shape. The integration is carried out numerically using stresses at 16 sampling points. For finding the maximum principal stresses all the three principal stresses at the points considered are scanned through. It is observed that except for two to three points near rim for shapes close to optimum, σ_θ stresses are maximum principal stresses.

(3) *Weighted objective function*

In this objective function, equal weightage is given to volume minimization and stress levelling. Thus

$$F(X) = 0.5\left[\frac{V}{V_0} + \frac{\oint(\sigma - \sigma_a)^2 \, ds}{\oint(\sigma_0 - \sigma_a)^2 \, ds}\right] \tag{11}$$

where V is the volume of rotating disk.

V_0 is the volume of the disk at initial shape $\oint(\sigma_0 - \sigma_a)^2 \, ds$ is stress levelling term for the initial shape.

The side constraints imposed are the lower limits on thicknesses at various selected points. In the present investigation, the minimum thickness is kept as 1% of the outer radius.

BRIEF DESCRIPTION OF AUTOMATED STRUCTURAL DESIGN PROGRAM SLPF

The authors have developed a general purpose structural synthesis program SLPF[8,9] for the shape optimization of continua. In this stress analysis is carried out by finite element analysis and optimization by improved move limit method of sequential linear programming[10]. The optimization is started with an initially supplied design variables. A user supplied subroutine generates a suitable mesh for the specified design variables. The stress analysis and the calculation of sensitivities of stresses to design variables are carried out in analyzer. With these values user supplied subroutines assemble the objective function, constraints and their sensitivities to design variables. Using these values the optimizer linearizes objective function and constraints and solves linear programming problem in the zone limited by the move limits on design variables. The optimization proceeds in the iterative manner from the new design point obtained by solving linear programming problem. The improvements in this method to overcome some of the difficulties encountered and to make the algorithm faster, as suggested in the earlier work[10], are incarporated. Except the mesh generation subroutines all other user supplied subroutines are very small. The program is made suitable for medium sized computers like ICL 1909. The program has been used for various shape optimization problems in the design of mechanical components[9].

CHOICE OF NON-DIMENSIONAL VARIABLES

In the present problem the objective function is the function of several parameter as shown below:

$$F(X) = \Phi(r_1, r_2, r_3, r_4, t_1, t_2, X_1, X_2, X_3, X_4, N, p) \quad (12)$$

where $r_1, r_2, r_3, r_4, t_1, t_2, X_1, X_2, X_3$, and X_4, are as defined earlier, N is rotating speed and p is the fit pressure.

Using outer radius of rim r_4 and unit weight of the material of disk 'ρ', for non-dimensionalising, the objective function can be expressed as

$$F(X) = \phi(r_1/r_4, r_2/r_4, r_3/r_4, t_1/r_4, t_2/r_4, X_1/r_4, X_2/r_4, X_3/r_4, X_4/r_4, N, p/(r_4\rho)) \quad (13)$$

The non-dimensionalised stress is expressed as $\sigma/(r_4\rho)$.

In the present study the first five terms in eqn (13) have been assumed to be fixed. The next four terms are the design variables and last two terms are considered for parametric studies.

PROGRESS OF OPTIMIZATION

For all the three objective functions considered, the optimization proceeds very smoothly. The optimum is reached in 45–75 min in ICL 1909. The optimum shapes, obtained with the three different objective functions for the same problem, are different. Figure 5 shows these shapes for a problem with $r_1/r_4 = 0.4$, $r_2/r_4 = 0.46$, $r_3/r_4 = 0.94$, $t_1/r = 0.16$, $t_2/r_4 = 0.05$, $N = 10000$ rpm, and $p = 0$. In the same figure, the variation of maximum principal stress along section A–A is shown. From these results the following observations can be made:

1. In the case of minimizing the difference between maximum and minimum tangential stress, the stress distribution obtained is good. At two points maximum stress is reached. The value of absolute maximum stress reached is the smallest among the optima obtained using different objective functions. But the shape obtained is not very satisfactory. There is ackward bulging near the hub.

2. The stress levelling can be achieved by decreasing the stresses at highly stressed region and by increasing stresses at low stressed region. From Fig. 5, it is clear that effort is mainly concentrated in increasing the stresses in the region of low stresses instead of reducing stresses in the highly stressed region. The reduction in volume is less.

3. The results obtained with weighted objective functions are very satisfactory. At optimum, the value of absolute maximum stress is very close to that in case of 1. Stress distribution is also satisfactory. Reduction in volume is large in comparison to the results with other two objective functions mentioned earlier. The weightage given to volume minimization in the objective function has helped in controlling the shape.

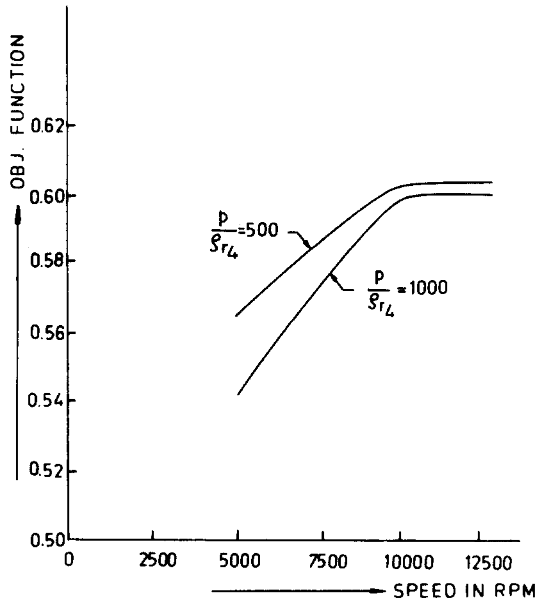

Fig. 6. Variation of objective function with speed.

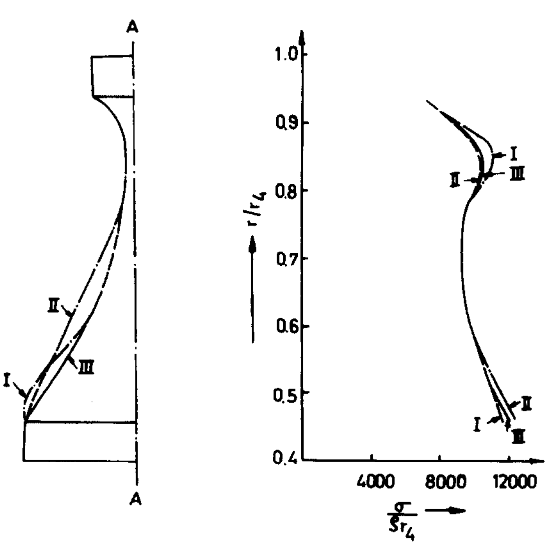

Fig. 5. Optimum shapes and corresponding stresses distribution for the three different objective functions.

RESULTS AND PARAMETRIC STUDIES

The parameters considered for detailed investigations are (1) the rotating speed and (2) fit pressure p. The four values of rotating speed considered are 5000 rpm, 7500 10000 and 12500 rpm. The different fit pressures considered are $p/(\rho r_4) = 0{,}500$ and 1000. In all these cases the other parameters are kept constant as shown below:

$$r_1/r_4 = 0.4 \quad r_2/r_4 = 0.46 \quad r_3/r_4 = 0.94$$
$$t_1/r_4 = 0.16 \quad t_2/r_4 = 0.05$$

When fit pressure is zero, no effect of speed is found on optimum shape. In all these cases, optimum design variables are (0.1210 0.0530 0.0190 0.0141). Figure 6 shows the variation of objective function with speed for $p/(r_4\rho) = 500$ and 1000. In Figs. 7 and 8, variation of optimum design variables with respect to speed is shown. It may be observed that as speed, effect of fit pressure is negligible and as in zero fit pressure case, a constant shape is reached. Comparison of design variables for different fit pressure indicate that as fit pressure increases thickening of the disk near the hub is required.

CONCLUSIONS

Optimum design of rotating disks are obtained by an accurate method which does not need any smoothening of the shape after optimization. An objective function

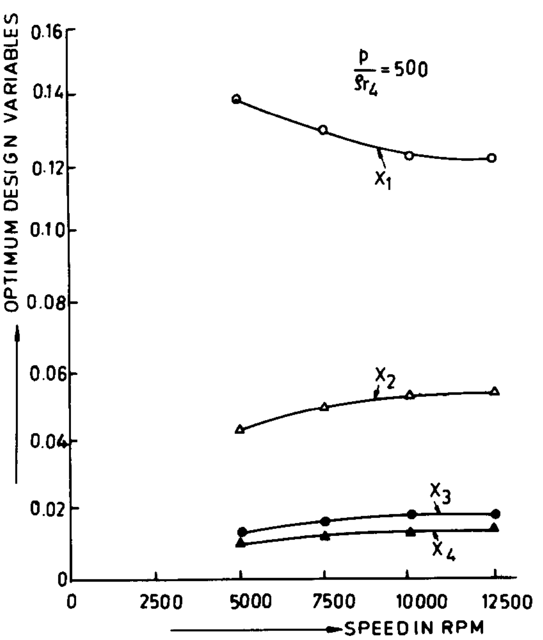

Fig. 7. Variation of optimum design variables with speed when $p/(\rho r_4) = 500$.

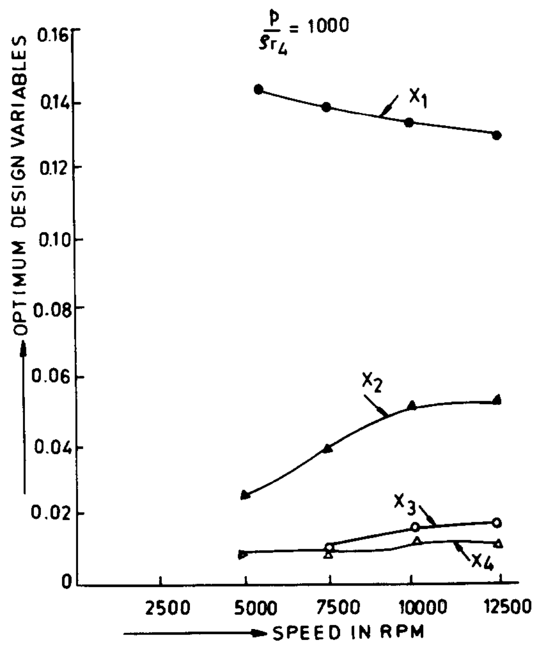

Fig. 8. Variation of optimum design variables with speed when $p/(\rho r_4) = 1000$.

with equal weightage for material volume and levelled stress distribution has been found to be very satisfactory. If fit pressure is not acting the speed has no effect on optimum shape. If fit pressure is acting lesser thickness is required with higher speed near hub. It needs thickening at points away from the hub. For a given speed, with higher fit pressure more thickening is necessary near the hub.

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