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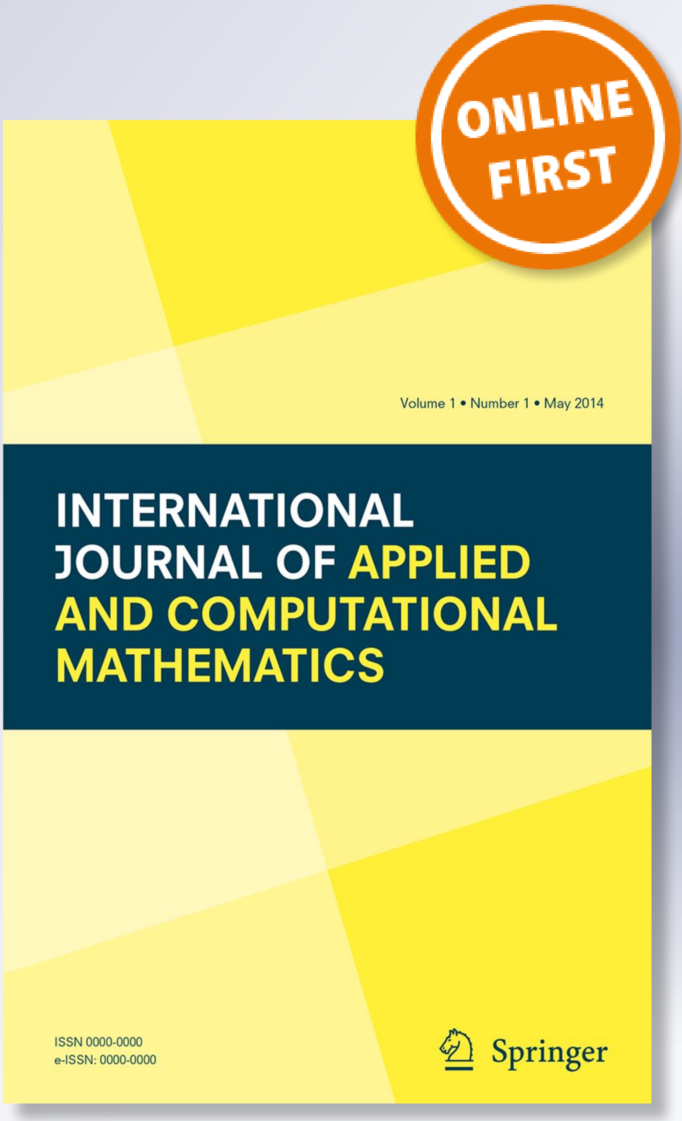

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ORIGINAL PAPER

On the Convergence of Stirling's Method for Fixed Points Under Not Necessarily Contractive Hypotheses

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Abstract Stirling's method is a useful alternative to Newton's method for approximating fixed points of nonlinear operators in a Banach space setting. This method has been studied under contractive hypotheses on the operator involved, thus limiting the applicability of it. In this study, we present a local as well as a semi-local convergence for this method based on not necessarily contractive hypotheses. This way, we extend the applicability of the method. Moreover, we present a favorable comparison of the new Kantorovich-type convergence criteria with the old ones using contractive hypotheses as well as with Newton's method. Numerical examples including Hammerstein nonlinear equations of Chandrasekar type appearing in neutron transport and in the kinetic theory of gases are solved to further illustrate the theoretical results.

Keywords Stirling's method · Newton's method · Local convergence · Semi-local convergence · Contractive hypotheses · Banach space

Mathematics Subject Classification 47H09 · 47H10 · 65G99 · 65H10 · 49M15

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Introduction

Let $G : D \subset B \to B$ be a Fréchet differentiable operator defined on a convex and open subset D of a Banach space B with values in B. A point p satisfying

$$G(p) = p \tag{1}$$

is called a fixed point of operator G. Many problems in Computational Sciences, such as problems in optimization, Applied Mathematics; Physics, Astrophysics; Biology; Economics; Chemistry, Control theory, Signal and image processing, inverse and ill-posed problems, least squares, engineering and other areas can be set up as equations of the type (1) using Mathematical modeling [1,2,9–11,14–18,23,24]. A fixed point p in closed form is desirable but usually it is unavailable. That is why most researchers rely on the generation of some iterative method approximating p. Picard's method or the method of successive substitutions [17] defined for all $n = 0, 1, 2, \ldots$ by $x_{n+1} = G(x_n)$, where $x_0 \in D$ is an initial point generates a sequence converging linearly to p provided that G is a contraction operator on D. To accelerate the convergence higher order methods have been proposed.

Stirling's method defined for all $n = 0, 1, 2, \ldots$ by

$$x_{n+1} = x_n - A_n^{-1}(x_n - G(x_n)), \tag{2}$$

where $x_0 \in D$ is an initial point and $A_n = I - G'(G(x_n))$ has been utilized to generate a sequence $\{x_n\}$ converging quadratically to p [7,20,22].

Stirling's method is a combination of Picard's method (or method of the successive substitutions) and Newton's method. Both methods utilize the same number of function evaluations per step, since each step utilizes the G and G'. These methods are compared in [7,20–22] under the crucial hypothesis that

$$\|G'(x)\| < 1 \tag{3}$$

in D. This is a major setback in the usage of Stirling's method. Let us consider the motivational example using function $G(x) = \frac{x^2}{2}$ for all $x \in D = [-1, 1]$. Then, we have that $\|F'(x)\| \leq 1$ for all $x \in D$, so (3) is violated. Hence, the results in [7,20–22] cannot guarantee the convergence of Stirling's method to p. However, we can show convergence although (3) is not satisfied (see the Examples). Notice also that Stirling's method is an important alternative in cases Newton's method fails to converge. In particular, Rall [22] used Banach theorem on fixed points in combination with Lipschitz continuous first Fréchet derivative for G' to provide semi-local convergence results for Stirling's method. Parhi and Gupta [20,21] studied the semi-local convergence of the method using Hölder continuous Fréchet-derivative. Another setback of the preceding approaches is the fact that the region of convergence is small in general. Therefore, extending the convergence region of iterative methods with additional or weaker hypotheses than before is very important in computational Mathematics and related areas.

"The novelty of the paper is the fact that convergence of Stirling's method is established, even if (3) is not satisfied. Moreover some other benefits are reported even if (3) is satisfied. The region of convergence for Stirling's method is extended using our new approach of restricted domains of convergence. Following this idea we find a subset of the original domain that also contains the Stirling's iterates leading to at least as tight Lipschitz conditions as in the earlier studies. The new constants are special cases of the old constants. Consequently no additional computational cost is required to obtain the improved convergence analysis. Furthermore, favorable comparisons are also reported with respect to Newton's method. The new ideas can be used to study other iterative methods using inverses of linear operators [1,2,2–21,23,24]".

The rest of the paper is designed as follows: In second section we report on the semi-local convergence; third section contains the local convergence. Fourth section contains the numerical examples.

Semi-local Convergence

We need an auxiliary results on majoring sequences for Stirling's method.

Lemma 2.1 [6] *Let $l_0 > 0,\ l > 0$ and $\eta > 0$ be parameters. Suppose*

$$h := K_0\eta \leq \frac{1}{2}, \tag{4}$$

where

$$K_0 = \frac{1}{8}\left(4l_0 + \sqrt{l_0 l} + \sqrt{l_0 l + 8l_0^2}\right). \tag{5}$$

Then scalar sequence $\{s_k\}$ defined by

$$\begin{aligned} s_0 = 0,\ s_1 = \eta,\ s_2 = \eta + \frac{l_0\eta^2}{2(1-l_0\eta)}, \\ s_{n+2} = s_{n+1} + \frac{l(s_{n+1}-s_n)}{2(l-l_0 s_{n+1})},\ \ for\ each\ n = 1, 2, \ldots \end{aligned} \tag{6}$$

is monotonically converging to each unique least upper bound s^*, which satisfies

$$\eta \leq s^* \leq q, \tag{7}$$

where

$$q = \eta + \frac{l_0\eta^2}{2(1-\alpha)(1-l_0\eta)} \tag{8}$$

and

$$\alpha = \frac{2l}{l + \sqrt{l^2 + 8l_0 l}}. \tag{9}$$

Let $U(w, \varrho)$ stand for an open ball with center $w \in B$ and of radius $\varrho > 0$. Moreover, let $\overline{U}(w, \varrho)$ be the closure of $U(w, \varrho)$.

The semi-local convergence analysis of Stirling's method is based on the hypotheses (H) for some $x_0 \in D$:

$(h_1) A_0^{-1} \in \mathcal{L}(B)$ and there exists $\eta \geq 0$ such that $\|A_0^{-1}(x_0 - G(x_0)\| \leq \eta$;
(h_2) $\|G(x) - G(x_0)\| \leq a_0\|x - x_0\|$, *for all* $x \in D$ *and some* $a_0 \geq 0$;
(h_3) $\|A_0^{-1}(G'(G(x_0)) - G'(y))\| \leq b_0\|G(x_0) - y\|$, *for all* $y \in D$ *and some* $b_0 \geq 0$;
(h_4) $\|A_0^{-1}(G'(u) - G'(v))\| \leq b\|u - v\|$ *for all* $u, v \in D_0 := D \cap U\left(x_0, \frac{1}{a_0 b_0}\right)$ *and some* $b \geq 0$;
(h_5) $\|G'(x)\| \leq c$ *for all* $x \in D_0$ *and some* $c \geq 0$;
(h_6) *Condition* (4) *holds for* $l_0 = b_0 \max\{3 + 2c, a_0\}$ *and* $l = b(3 + 2c)$;
(h_7) $\overline{U}(x_0, R) \subseteq D$, *where* $R := \|x_0 - G(x_0)\| + a_0 s^*$, *where* s^* *is given in Lemma* 2.1.

Next, we present the semi-local convergence analysis of Stirling's method using (H) hypotheses and the preceding notation.

Theorem 2.2 *Suppose that the hypotheses (H) hold. Then, sequence* $\{x_n\}$ *starting from* x_0 *and generated by Stirling's method exists and converges to a fixed point* $p \in \overline{U}(x_0, s^*)$ *of operator G so that*

$$\|x_n - x_p\| \leq r^* - r_{n-1},$$

where, scalar sequence $\{r_n\}$ *is defined by*

$$\begin{aligned} &r_0 = 0,\ r_1 = \eta,\ r_2 = r_1 + \tfrac{b_0(3+2c)(r_1-r_0)^2}{2(1-a_0 b_0 r_1)}, \\ &r_{n+2} = r_{n+1} + \tfrac{b(3+2c)(r_{n+1}-r_n)^2}{2(1-a_0 b_0 r_{n+1})} \ \textit{for all}\ n = 1, 2, \ldots \end{aligned} \tag{10}$$

and $r^* = \lim_{n\to\infty}$.

Proof A simple induction argument shows that

$$r_n \leq s_n \leq s^* \tag{11}$$

and

$$r_n \leq r_{n+1} \tag{12}$$

$$r_{n+1} - r_n \leq s_{n+1} - s_n \tag{13}$$

by the choice of l_0, l, (6) and (10). Hence, sequence $\{r_n\}$ is nondecreasing and bounded above by s^* and as such it converges to its unique least upper bound r^*. Next, we shall show that sequence $\{r_n\}$ majorizes $\{x_n\}$. That is, we shall show that for each $m = 0, 1, 2, \ldots$

$$\|x_m - x_{m-1}\| \leq r_m - r_{m-1} \tag{14}$$

and

$$\overline{U}(x_m, r^* - r_m) \subseteq \overline{U}(x_{m-1}, r^* - r_{m-1}). \tag{15}$$

Let $w \in \overline{U}(x_1, r^* - r_1)$. We get:

$$\|w - x_0\| \leq \|w - x_1\| + \|x_1 - x_0\| \leq r^* - r_1 + r_1 - r_0 = r^* - r_0,$$

so $w \in \overline{U}(x_0, r^* - r_1)$. By (h_1) and (10) we have

$$\|x_1 - x_0\| = \|A_0^{-1}(x_0 - G(x_0)\| \leq \eta = r_1 = r_1 - r_0,$$

so (14) and (15) hold for $m = 0$. Suppose (14) and (15) hold for all integers smaller or equal to m. Then, we get

$$\|x_m - x_0\| \leq \sum_{i=1}^{m} \|x_i - x_{i-1}\| \leq \sum_{i=1}^{m} (r_i - r_{i-1}) = r_m - r_0 = r_m \leq r^*, \tag{16}$$

and

$$\|x_m + \theta(x_m - x_0) - x_0\| \le r_{m-1} + \theta(r_m - r_{m-1}) \le r^* \tag{17}$$

for all $\theta \in [0, 1]$. Let $x \in D_0$, then

$$\begin{aligned} \|x_0 - G(x)\| &\le \|x_0 - G(x_0)\| + \|G(x) - G(x_0)\| \\ &\le \|x_0 - G(x_0)\| + a_0\|x_0 - x_1\| \le \|x_0 - G(x_0)\| + a_0 r^* = R. \end{aligned} \tag{18}$$

Using the induction hypotheses, (h_2), (h_3) and (16), we have

$$\begin{aligned} &\|A_0^{-1}(G'(G(x_0)) - G'(G(x_n)))\| = \|A_0^{-1}(G'(G(x_0)) - G'(G(x_m)))\| \\ &\le b_0\|G(x_0) - G(x_m)\| = \|b_0 a_0\|x_m - x_0\| \le b_0 a_0 r_m < 1, \end{aligned} \tag{19}$$

so by the Banach perturbation lemma [14] and (19) A_m^{-1} exists and

$$\|A_m^{-1}A_0\| \le (1 - b_0 a_0\|x_m - x_0\|)^{-1} \le (1 - b_0 a_0 r_m)^{-1}. \tag{20}$$

Using Stirling's method we obtain in turn the approximation

$$\begin{aligned} A_0(x_m - G(x_m)) &= A_0(x_m - G(x_{m-1}) - G(x_m) + G(x_{m-1})) \\ &= A_0(G'(G(x_{m-1}))(x_m - x_{m-1}) - G(x_m) + G(x_{m-1})) \\ &= A_0 \int_{x_{m-1}}^{x_m} (G'(G(x_{m-1})) - G'(x))dx \\ &= A_0 \int_0^1 (G'(G(x_{m-1})) - G'(x_{m-1}) \\ &\quad + \theta(x_m - x_{m_1}))(x_m - x_{m-1})d\theta. \end{aligned} \tag{21}$$

In view of (h_4), (h_5), (14) and (21), we obtain in turn

$$\begin{aligned} \|A_0(x_m - G(x_m))\| &\le \int_0^1 \bar{b}\|G(x_{m-1}) - x_{m-1} - \theta(x_m - x_{m_1})\|\|x_m - x_{m-1}\|d\theta \\ &\le \bar{b}\int_0^1 (\|G(x_{m-1}) - x_{m-1}\| + \theta\|x_m - x_{m_1}\|)\|x_m - x_{m-1}\|d\theta \\ &\le \bar{b}\int_0^1 (\|I - G'(G(x_{m-1}))\| + \theta)\|x_m - x_{m-1}\|^2 d\theta \\ &\le \frac{\bar{b}(3+2c)}{2}\|x_m - x_{m-1}\|^2 \le \frac{\bar{b}(3+2c)}{2}(r_m - r_{m-1})^2, \end{aligned} \tag{22}$$

where

$$\bar{b} = \begin{cases} b_0, & m = 1 \\ b & m > 1. \end{cases}$$

Moreover, by Stirling's method, (10), (14), (20) and (22), we get that

$$\begin{aligned} \|x_{m+1} - x_m\| &= \|A_m^{-1}(x_m - G(x_m))\| \le \|A_m^{-1}A_0\|\|A_0^{-1}(x_m - G(x_m))\| \\ &\le \frac{\bar{b}(3+2c)(r_m - r_{m-1})^2}{2(1 - a_0 b_0 r_m)} = r_{m+1} - r_m, \end{aligned} \tag{23}$$

so the induction for (14) is completed. □

Furthermore, for $w \in \overline{U}(x_{m+1}, r^* - r_{m+1})$, we have

$$\|w - x_{m+1}\| \le \|w - x_{m+1}\| + \|x_{m+1} - x_m\| \le r^* - r_{m+1} + r_{m+1} - r_m = r^* - r_m,$$

which completes the induction for (15).

We have that that sequence $\{r_n\}$ is complete. By (1.11) and (15) sequence $\{x_m\}$ is complete too in a Banach space B, so there exists $p \in \overline{U}(x_0, r^*)$ such that $\lim_{m\to\infty} = p$. By letting $m \to +\infty$ in (12), the estimate $\|A_0^{-1}(x_m - G(x_m))\| \le \frac{\bar{b}}{2}(3 + 2c)(r_m - r_{m-1})^2$, we get $p = G(p)$. Finally, estimate (13) follows from (14) using standard majorization procedures [2,3,8,14].

Remark 2.3 (a) If $a_0 \in [0, 1)$ and $r^* \geq \|x_0 - F(x_0)\|/(1 - a_0)$, then these conditions can replace (h_7). In particular, in view of the estimate

$$\begin{aligned} \|x_0 - F(x)\| &= \|x_0 - F(x_0) + F(x_0) - F(x)\| \\ &\leq \|x_0 - F(x)\| + a_0\|x - x_0\| \leq \|x_0 - F(x_0)\| + a_0 \leq r^* \end{aligned}$$

for all $x \in D$. Hence, (h_6) is replaced by (h_7)' $\overline{U}(x_0, r^*) \subseteq D$ provided that $a_0 \in [0, 1)$ and $r^* \geq \|x_0 - F(x_0)\|/(1 - a_0)$.

(b) Limit point r^* can be replaced in Theorem 1 by q given in closed form.

(c) It follows from (10), (14) and (19) that the Q-order of Stirling's method is 2.

Concerning the uniqueness of the fixed point p, we have:

Proposition 2.4 *Under the hypotheses (H), further suppose that there exists $R^* \geq r^*$ such that*

$$b_0((2a_0 + 1)r^* + R^*) < 2, \tag{24}$$

then, p is the only fixed point of G in $D_1 := D \cap \overline{U}(x_0, R^*)$.

Proof Let $p^* \in D_1$ with $G(p^*) = p^*$. The existence of p has been established in Theorem 1 under the hypotheses (H). Let $T : B \to B$ defined by

$$T = \int_0^1 A_0^{-1} G'(p + \theta(p^* - p))d\theta.$$

Using (h_2), (h_3) and (24), we have in turn that

$$\begin{aligned} \|I - (A_0^{-1} - I)\| &= \| \textstyle\int_0^1 A_0^{-1}(G'(p + \theta(p^* - p)) - G'(G(x_0)))d\theta\| \\ &\leq b_0 \textstyle\int_0^1 \|p + \theta(p^* - p) - G(x_0)\|d\theta \\ &\leq b_0 \textstyle\int_0^1 \|G(p) - G(x_0) + \theta(p^* - x_0) - \theta(p - x_0)\|d\theta \\ &\leq b_0 \left[a_0 r^* + \tfrac{1}{2}(R^* + r^*)\right] < 1, \end{aligned} \tag{25}$$

so $(A_0^{-1} - T)^{-1}$exists. Then, from the identity

$$0 = A_0^{-1}[p^* - G(p^*) - p + G(p)] = (A_0^{-1} - T)(p^* - p), \tag{26}$$

we conclude that $p = p^*$. □

Remark 2.5 Let $L_0 = \max\{a_0 b_0, b(3 + 2c)\}$. Then we can define sequence $\{t_n\}$ by

$$t_0 = 0, \;\; t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)} = t_n + \frac{L_0(t_n - t_{n-1})^2}{2(1 - L_0 t_n)}, \tag{27}$$

where

$$f(t) = \frac{L_0}{2}t^2 - t + \eta. \tag{28}$$

Suppose

$$h = L_0\eta \leq \frac{1}{2}. \tag{29}$$

Sequence $\{t_n\}$ looks like the usual majorizing sequence appearing in the semi-local convergence of Newton's method, whereas (29) is the corresponding Newton-Kantorovich sufficient semi-local convergence criterion for Newton's method. Clearly, $\{t_n\}$, (29) can replace $\{r_n\}$, (4) in Theorem 1, since a simple inductive arguments shows that

$$\begin{aligned} r_n &\le t_n \\ r_{n+1} - r_n &\le t_{n+1} - t_n \\ r^* &\le t^* = \lim_{n\to\infty} t_n = \frac{1-\sqrt{1-2L_0\eta}}{L_0} \end{aligned}$$

and $h \le 1/2 \Rightarrow h_0 \le 1/2$.

There is a plethora of error bounds for sequence $\{t_n\}$ [1–24]. A direct comparison between Newton's and Stirling's method is possible under the above setting but we must use

$$\begin{aligned} \|(I-G'(x_0))^{-1}(G'(x)-G'(x_0))\| &\le \bar{b}_0\|x-x_0\|, \; x \in D \\ \|(I-G'(x_0))^{-1}(G'(x)-G'(y))\| &\le \bar{b}\|x-y\|, \; x \in D_5 = D \cap U\left(x_0, \frac{1}{\bar{b}_0}\right) \\ \|(I-G'(x_0))^{-1}(G'(x)-G'(x_0))\| &\le \bar{\eta} \end{aligned}$$

and the corresponding criterion, iteration $\{\bar{t}_n\}$ are [6]

$$H = L\bar{\eta} \le \frac{1}{2} \tag{30}$$

where

$$L = \frac{1}{8}\left(4\bar{b}_0 + \sqrt{\bar{b}_0\bar{b}} + \sqrt{\bar{b}_0\bar{b} + 8\bar{b}_0^2}\right)$$

and

$$\begin{aligned} \bar{t}_0 = 0,\; \bar{t}_1 = \bar{\eta},\; \bar{t}_2 &= \bar{t}_1 + \frac{\bar{b}_0(\bar{t}_1-\bar{t}_0)^2}{2(1-\bar{b}_0\bar{t}_1)}, \\ \bar{t}_{n+2} &= \bar{t}_{n+1} + \frac{\bar{b}(\bar{t}_{n+1}-\bar{t}_n)^2}{2(1-\bar{b}\bar{t}_{n+1})}. \end{aligned}$$

Local Convergence

The local convergence analysis of Stirling's method is based on the hypotheses (C):

(c_1) *There exists* $p \in D$ *with* $p = G(p)$ *such that the inverse of* $A_\star = I - G'(p)$ *exists*;
(c_2) $\|G(x) - G(p)\| \le \beta_0\|x-p\|$, *for all* $x \in D$ *and some* $\beta_0 \ge 0$;
(c_3) $\|A_\star^{-1}(G'(p) - G'(x))\| \le \gamma\|p-x\|$, *for all* $x \in D$ *and some* $\gamma \ge 0$;
(c_4) $\|A_\star^{-1}(G'(x) - G'(y))\| \le d\|x-y\|$ *for all* $x, y \in D_2 := D \cap U\left(p, \frac{1}{\beta_0\gamma}\right)$ *and some* $d \ge 0$;
(c_5) $\overline{U}(p, \varrho^*) \subseteq D$,

where

$$\varrho^* = \frac{2}{2\beta_0\gamma + (1+2\beta_0)d}. \tag{31}$$

Theorem 3.1 *Suppose that the hypotheses (C) hold. Then, Stirling's sequence $\{x_n\}$ starting from $x_0 \in U(p, \varrho^*)$ converges to p so that*

$$\|x_{n+1} - x^*\| \leq \frac{d(1+2\beta_0)}{2(1-\beta_0\gamma\|x_n - p\|)}\|x_n - p\|^2. \tag{32}$$

Proof We use induction and the estimates for $x_m \in U(p, \varrho^*)$

$$\begin{aligned}\|A_\star^{-1}(A_\star - A(x_m))\| &= \|A_\star^{-1}(G'(G(p)) - G'(G(x_n)))\| \\ &\leq \gamma\|G(p) - G(x_m)\| \leq b_0\gamma\|p - x_m\| \\ &\leq \beta_0\gamma\varrho^* < 1,\end{aligned} \tag{33}$$

and

$$\begin{aligned}\|G(x_n) - x_m\| &\leq \|(G(x_m) - p) + p - x_m\| \\ &\leq \|G(x_m) - G(p)\| + \|x_m - p\| \\ &\leq (\beta_0 + 1)(\|x_m - p\|,\end{aligned} \tag{34}$$

and

$$\begin{aligned}&\|A_\star^{-1}(G(x_m) - G(p) - G'(G(x_m))(x_m - p))\| \\ &\quad = \|\textstyle\int_0^1 A_\star^{-1}[G'(\theta x_m + (1-\theta)p) - G'(\theta F(x_m) + (1-\theta)F(x_m))](x_m - p)d\theta\| \\ &\quad \leq d\textstyle\int_0^1[\theta\|x_m - G(x_m)\| + (1-\theta)\|p - G(x_m)\|]\|x_m - p\|d\theta \\ &\quad = \frac{d}{2}[\|x_m - F(x_m)\| + \|G(x_m) - p\|] \leq \frac{d(1+2\beta_0)}{2}\|x_m - p\|^2\end{aligned} \tag{35}$$

so by (33) and (35), we arrive at (32). □

Concerning the uniqueness part:

Proposition 3.2 *Suppose that the hypotheses (C) hold for $\beta_0 \in [0, 1)$. Then, p is the unique fixed point of operator G on $\overline{U}(p, \varrho^*)$.*

Proof We have by (c_2) that for $p \neq p^* \in \overline{U}(p, \varrho^*)$ with $p^* = G(p^*)$:

$$\|p - p^*\| = \|G(p) - G(p^*)\| \leq \beta_0\|p - p^*\| < \|p - p^*\|,$$

so we deduce that $p = p^*$. □

Remark 3.3 Condition (c_2) does not imply that G' is a contraction operator. Hence, the applicability of Stirling's method is expanded in the local convergence case too. The local convergence results in [22, see, e.g, Theorem 4, pp 16] given in non-affine invariant form can be improved under the same hypotheses.

Proposition 3.4 *Suppose: There exists $\delta \geq 0$ and $\mu \in [0, 1)$, $\mu_0 \in [0, 1]$ such that*

$$\begin{aligned}\|G'(x)\| &\leq \mu; \\ \|G'(x) - G'(y)\| &\leq \delta\|x - y\| \ for\ all\ x, y \in D; \\ \|G(x) - G(p) &\leq \mu_0\|x - p\| \text{ for all } \ x \in D\end{aligned}$$

and

$$U(p, \varrho_1^*) \subseteq D,$$

where

$$\varrho_1^* = \frac{2(1-\mu)}{\delta(1+2\mu_0)}. \tag{36}$$

Then, Stirling's method starting from $x_0 \in U(p, \varrho^*) - \{p\}$ *converges to* p *so that*

$$\|x_{n+1} - p\| \leq \frac{\delta(1 + 2\mu_0)}{2(1 - \mu)} \|x_n - p\|^2. \tag{37}$$

Proof Follow the proof of Theorem 4 in [22] but use

$$\|G(p) - G(x_m)\| \leq \alpha_0 \|p - x_m\|$$

instead of $\|G(p) - G(x_m)\| \leq \mu\|p - x_m\|$. □

Proposition 3.5 *The radius of convergence and corresponding error bounds in [22] are given, respectively by*

$$\varrho_0^* = \frac{2(1 - \mu)}{\delta(1 + 2\mu)} \tag{38}$$

and

$$\|x_{n+1} - p\| \leq \frac{\delta(1 + 2\mu)}{2(1 - \mu)} \|x_n - p\|^2. \tag{39}$$

Our results (36) and (37) improve (38) and (39), respectively, since $\mu_0 \leq \mu$. These advantages require the same computational effort as in [7,20–22], since in practice the computation of a requires the computation of a_0 as a special case.

Numerical Examples

We present three numerical examples to show that out results can be used to solve equations but earlier ones using even stronger contractivity type hypotheses cannot be used. The first example involves the local convergence and the last two the semi-local convergence of Stirling's method.

Example 4.1 Let $B = \mathbb{R}^3$, $D = \bar{U}(0, 1)$ and $p = (0, 0, 0)^T$. Define function F on D for $w = (x, y, z)^T$ by $G(w) = w + F(w)$, where

$$F(w) = \left(e^x - 1, \frac{e - 1}{2}y^2 + y, z\right)^T.$$

Then, the Fréchet-derivative is given by

$$G'(w) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e - 1)y + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + I.$$

Notice that using the (C) conditions, we get $\beta_0 = e + 1$, $\gamma = e - 1$, $d = e^{\frac{1}{\beta_0 \gamma}}$, $\|A_*^{-1}\| = 1$ and the radius ρ^* is given by $\rho^* = 0.0883232487593$. Notice that earlier results [7,20–22] cannot apply, since G is not a contraction on D.

Example 4.2 Looking again at the motivational example at the introduction of this study, let $x_0 = 0.1$. Then hypotheses $(\mathcal{H})$ are satisfied for $a_0 = 0.505$, $A_0 = 0.99995$, $A_0^{-1} = 1.00005$, $\eta = 0.0099504$, $b_0 = b = A_0^{-1}$, $c = 1$, $l_0 = l = 5A_0^{-1}$, $K_0 = l$, since (4) gives $l\eta = 0.0497522 < 0.5$ and $R = 0.02$. Hence,our Theorem 3.1 guarantees the convergence of Stirling's method to $p = 0$ starting at x_0.

Example 4.3 Let $B = C[0,1]$, $D = \bar{U}(p,1)$ and consider the nonlinear integral equation of the mixed Hammerstein-type [4,7] defined by

$$x(s) = \lambda(s) + \varepsilon \int_0^1 K(s,t)x(t)^2 dt, \;\; s \in [0,1], \; x \in B, \tag{40}$$

where the kernel P is a continuous function defined on the interval $[0,1] \times [0,1]$, ε is a constant and $\lambda(s)$ is continuous on $[0,1]$. Equations of the form (40) appear in the kinetic theory of gases or neutron transport [2,3,14]. Equation (40) can be written as

$$G(x) = x \tag{41}$$

where $G : C[0,1] \longrightarrow C[0,1]$ is defined by

$$G(x)(s) = \lambda(s) + \varepsilon \int_0^1 K(s,t)x(t)^2 dt \tag{42}$$

The Fréchet-derivative of G is defined by

$$G'(x)v(s) = 2\varepsilon \int_0^1 K(s,t)v(t)dt. \tag{43}$$

We have $\|G'(x)\| \leq 2|\varepsilon|\|\tilde{d}\|\|x\|, \; \|G'(G(x))\| \leq 2|\varepsilon|\|\tilde{d}\|\|G(x)||$, where $\tilde{d} = \max_{s\in[0,1]} |\int_0^1 K(s,t)dt|$. Choose $x_0 = x_0(s) = 1$, $\lambda(s) = \mu - \varepsilon s$ and $K(s,t) = s$ for all $(s,t) \in [0,1] \times [0,1]$ and some $\mu \in \mathbb{R}$. Then, we have $\tilde{d} = 1$, $\eta = \frac{|1-\mu|}{1-2|\mu||\varepsilon|}$, $c = a_0 = 4|\varepsilon|$ and $K_0 = l_0 = l = \frac{3+8|\varepsilon|}{1-2|\mu||\varepsilon|}$ provided that $2|\mu||\varepsilon| < 1$. If $4|\varepsilon| \geq 1$, then condition (3) is not satisfied. Hence, the earlier results [7,20–22] cannot apply to solve (41).

However, our condition (4) becomes

$$\frac{2(3+8|\varepsilon|)|1-\mu||\varepsilon|}{(1-2|\mu\varepsilon|)^2} \leq \frac{1}{2}. \tag{44}$$

Choose $\varepsilon = 0.255$ and $\mu = 0.95$. Then condition (3) is not satisfied but (44) is satisfied, since $0.4836302 < 0.5$ and $2|\mu||\varepsilon| = 0.4845 < 1$. Hence, the conclusion of Theorem 3.1 apply.

Conclusion

The local as well as the semi-local convergence of Stirling's method is studied under not necessarily contractive hypotheses on G or G' for approximating a fixed point p of operator G in a Banach space setting. Our work extends the applicability of this method, since in all earlier studies [7,20–22] very restrictive contractive condition (3) was assumed. Numerical examples where earlier works cannot apply to solve equations but our work applies are also presented in this study in both the local as well as the semi-local convergence case.

References

1. Amat, S., Busquier, S., Negra, M.: Adaptive approximation of nonlinear operators. Numer. Funct. Anal. Optim. **25**, 397–405 (2004)
2. Argyros, I.K.: Computational theory of iterative methods. In: Chui, C.K., Wuytack, L. (eds.) Series: Studies in Computational Mathematics, vol. 15. Elsevier Publication Co., New York, USA (2007)
3. Argyros, I.K.: Convergence and Application of Newton-type Iterations. Springer, Berlin (2008)

4. Argyros, I.K.: A semi-local convergence analysis for directional Newton methods. Math. Comput. **80**, 327–343 (2011)
5. Argyros, I.K.: A unified local-semi-local convergence analysis and applications for two-point Newton-like methods in Banach space. J. Math. Anal. Appl. **298**, 374–397 (2004)
6. Argyros, I.K., Hilout, S.: Weaker conditions for the convergence of Newton's method. J. Complex. **28**, 364–387 (2012)
7. Argyros, I.K.: Stirling's method and fixed points of nonlinear operator equations in Banach spaces. Bull. Inst. Math. Acad. Sin. (N.S.) **23**, 13–20 (1995)
8. Cianciaruso, F., De Pascale, E.: Estimates of majorizing sequences in the Newton–Kantorovich method: a further improvement. J. Math. Anal. Appl. **322**, 329–335 (2006)
9. Cătinas, E.: The inexact, inexact perturbed, and quasi-Newton methods are equivalent models. Math. Comput. **74**, 291–301 (2005)
10. Deuflhard, P.: Newton Methods for Nonlinear Problems. Affine Invariance and Adaptive Algorithms, Springer Series in Computational Mathematics, Vol. 35. Springer, Berlin (2004)
11. Ezquerro, J.A., Hernández, M.A.: On the R-order of convergence of Newton's method under mild differentiability conditions. J. Comput. Appl. Math. **197**, 53–61 (2006)
12. Ezquerro, J.A., Gutiérrez, J.M., Hernández, M.A., Romero, N., Rubio, M.J.: The Newton method: from Newton to Kantorovich. Gac. R. Soc. Mat. Esp. **13**, 53–76 (2010). (Spanish)
13. Gutiérrez, J.M., Hernández, M.A., Romero, N.: Newton-type methods of high order and domains of semi-local and global convergence. Appl. Math. Comput. **214**, 142–154 (2009)
14. Kantorovich, L.V., Akilov, G.P.: Functional Analysis. Pergamon Press, Oxford (1982)
15. Magreñán, A.A.: Different anomalies in a Jarratt family of iterative root finding methods. Appl. Math. Comput. **233**, 29–38 (2014)
16. Magreñán, A.A.: A new tool to study real dynamics: The convergence plane. Appl. Math. Comput. **248**, 29–38 (2014)
17. Proinov, P.D.: New general convergence theory for iterative processes and its applications to Newton–Kantorovich type theorems. J. Complexity **26**, 3–42 (2010)
18. Păvăloiu, I.: Introduction in the Theory of Approximation of Equations Solutions. Editura Dacia, Cluj-Napoca, Romania (1976)
19. Potra, F.A.: On the convergence of a class of Newton-like methods. Iterative solution of nonlinear systems of equations (Oberwolfach 1982). Lecture Notes in Mathematics, vol. 953, pp. 125–137. Springer, Berlin (1982)
20. Parhi, S.K., Gupta, D.K.: Semilocal convergence of Stirling's method under Hölder continuous first derivative in Banach spcaes. Int. J. Comput. Math. **87**, 2752–2759 (2010)
21. Parhi, S.K., Gupta, D.K.: Relaxing convergence conditions for Stirling's method. Math. Methods Appl. Sci. **33**, 224–232 (2010)
22. Rall, L.B.: Convergence of Stirling's method in Banach spaces. Aequ. Math. **12**, 12–20 (1975)
23. Rheinboldt, W.C.: An adaptive continuation process for solving systems of nonlinear equations. In: Tikhonov A.N. et al. (eds.) Mathematical Models and Numerical Methods, **3**(19), 129–142. Banach Center, Warsaw
24. Traub, J.F., Woźniakowsi, H.: Convergence and complexity of Newton iteration for operator equations. J. Assoc. Comput. Mach. **26**, 250–258 (1979)