Applied Mathematics E-Notes, 2(2002), 192-197 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

On Graceful Trees *

Suresh Manjanath Hegde†, Sudhakar Shetty‡

Received 30 October 2001

Abstract

A (p, q)-graph $G = (V, E)$ is said to be (k, d)-*graceful*, where k and d are positive integers, if its p vertices admits an assignment of a labeling of numbers $0, 1, 2, ..., k + (q - 1)d$ such that the values on the edges defined as the absolute difference of the labels of their end vertices form the set $\{k, k + d, ..., k + (q - 1)d\}$. In this paper we prove that a class of trees called T_P-trees and subdivision of T_P-trees are (k, d)-graceful for all positive integers k and d.

1 INTRODUCTION

For all terminology and notation in graph theory we follow Harary [5].

Graphs labeling, where the vertices are assigned values subject to certain conditions, have often been motivated by practical problems. Labeled graphs serves as useful mathematical models for a broad range of applications such as Coding theory, including the design of good radar type codes, synch-set codes, missile guidance codes and convolution codes with optimal autocorrelation properties. They facilitate the optimal nonstandard encoding of integers. A systematic presentation of diverse applications of graph labelings is presented in [3].

Given a graph $G = (V, E)$, the set N of non-negative integers and a commutative binary operation $* : N \times N \to N$, every vertex function $f : V(G) \to N$ induces an edge function $g_f : E(G) \to N$ such that $g_f(uv) = f(u) * f(v)$ for all $uv \in E(G)$.

A function f is called a *graceful labeling* of a (p, q)-graph $G = (V, E)$ if f is an injection from the vertices of G to the set $\{0, 1, 2, ..., q\}$ such that, when each edge uv is assigned the label $|f(u) - f(v)|$, the resulting edge labels are distinct. Rosa [6] introduced this concept in 1967 and also defined a *balanced labeling* of a graph G is a graceful labeling f of G such that for each edge uv of G either $f(u) \leq c < f(v)$ or $f(v) \leq c < f(u)$ for some integer c, called characteristic of f. The Ringel-Kotzig conjecture that all trees are graceful has been the focus of many papers [4].

Acharya and Hegde [2] generalized *graceful labeling* to (k, d)-*graceful labeling* by permitting the vertex labels to belong to $\{0, 1, 2, ..., k + (q - 1)d\}$ and requiring the set of edge labels induced by the absolute difference of labels of adjacent vertices to be

*Mathematics Subject Classifications: 05C78

†Department of Mathematical and Computational Sciences, Karnataka Regional Engineering College, Surathkal, Srinivasanagar, 574157, Karnataka, India

‡Department of Mathematics, N. M. A. M. Institute Of Technology, Nitte 574110, Karnataka, India

$\{k, k+d, , ..., k+(q-1)d\}$, where k and d are positive integers. They also introduce an analog of *balanced labeling* , a (k,d)-*balanced labeling* of a graph G is a (k,d)-*graceful labeling* f of G such that for each edge uv of G either $f(u) \leq c < f(v)$ or $f(v) \leq c < f(u)$ for some integer c. One can note that $(1,1)$-*graceful labeling* and *graceful labeling* are identical.

In this paper we prove that a class of trees called T_P-trees*(transformed trees)* and subdivision $S(T)$ of a T_P-tree T, obtained by subdividing every edge of T exactly once are (k,d)-graceful for all positive integers k and d.

2 TRANSFORMED TREES (T_P-TREES)

Let T be a tree and u_o and v_o be two adjacent vertices in T. Let there be two pendant vertices u and v in T such that the length of $u_o - u$ path is equal to the length of $v_o - v$ path. If the edge u_ov_o is deleted from T and u, v are joined by an edge uv, then such a transformation of T is called an *elementary parallel transformation* (or an *ept*) and the edge u_ov_o is called a *transformable edge* (Acharya [1]).

If by a sequence of *ept*'s T can be reduced to a path then T is called a T_P-*tree*(*transformed tree*) and any such sequence regarded as a composition of mappings (ept's) denoted by P, is called a *parallel transformation of T*. The path, the image of T under P is denoted as $P(T)$.

A T_P-tree and a sequence of two ept's reducing it to a path are illustrated in *Fig*-1.

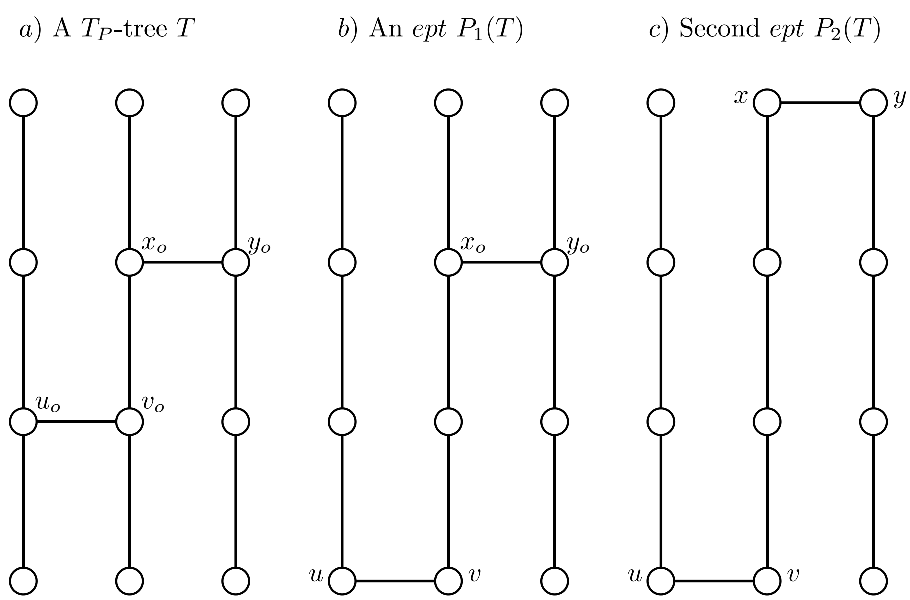

Fig-1: A T_P-tree and a sequence of two *ept*'s reducing it to a path.

THEOREM 1. Every T_P-tree is (k,d)-*graceful* for all positive integers k and d.

PROOF. Let T be a T_P-tree with $n+1$ vertices. By the definition of a T_P-tree there exists a parallel transformation P of T such that for the path $P(T)$ we have (i) $V(P(T)) = V(T)$ and (ii) $E(P(T)) = (E(T) - E_d) \cup E_P$, where E_d is the set of edges deleted from T and E_P is the set of edges newly added through the sequence $P = (P_1, P_2, ..., P_k)$ of the *epts* P used to arrive at the path $P(T)$. Clearly E_d and E_P have the same number of edges.

Now denote the vertices of $P(T)$ successively as $v_1, v_2, v_3, ..., v_{n+1}$ starting from one pendant vertex of $P(T)$ right up to other. The labeling f defined by

$$f(v_i) = \begin{cases} k + (q-1)d - [(i-1)/2]d & \text{for odd } i, \quad 1 \le i \le n+1 \\ [(i/2) - 1]d & \text{for even } i, \quad 2 \le i \le n+1 \end{cases}$$

where k and d are positive integers and q is the number of edges of T, is a (k,d)-*graceful* labeling of the path $P(T)$.

Let $v_iv_j$be an edge in T for some indices i and j, $1 \le i < j \le n+1$ and let P_1 be the *ept* that deletes this edge and adds the edge $v_{i+t}v_{j-t}$where t is the distance of v_i from v_{i+t} as also the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1as one of the constituent *epts*.

Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$ it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$. Therefore i and j are of opposite parity, i.e., i is odd and j is even or vice-versa.

The value of the edge v_iv_j is given by,

$$g_f(v_iv_j) = g_f(v_iv_{i+2t+1}) = |f(v_i) - f(v_{i+2t+1})|\,. \tag{1}$$

If i is odd and $1 \le i \le n$, then

$$\begin{aligned} f(v_i) - f(v_{i+2t+1}) &= k + (q-1)d - [(i-1)/2]d - [((i+2t+1)/2) - 1]d \\ &= k + (q-1)d - (i+t-1)d. \end{aligned} \tag{2}$$

If i is even and $2 \le i \le n$, then

$$\begin{aligned} f(v_i) - f(v_{i+2t+1}) &= [(i/2) - 1]d - [k + (q-1)d] + [(i+2t+1-1)/2]d \\ &= (i+t-1)d - [k + (q-1)d]. \end{aligned} \tag{3}$$

Therefore from (1), (2) and (3),

$$g_f(v_iv_j) = |k + (q-1)d - (i+t-1)d|,\ 1 \le i \le n. \tag{4}$$

Now

$$\begin{aligned} g_f(v_{i+t}v_{j-t}) &= g_f(v_{i+t}v_{i+t+1}) = |f(v_{i+t}) - f(v_{i+t+1})| \\ &= |k + (q-1)d - (i+t-1)d|,\ 1 \le i \le n. \end{aligned} \tag{5}$$

Therefore from (4) and (5)

$$g_f(v_iv_j) = g_f(v_{i+t}v_{j-t}).$$

Hence f is a (k,d)-*graceful* labeling of T_P-tree T. The proof is complete.

For example, a $(1,1)$-*graceful* labeling of a T_P-tree T using Theorem 1, is shown in *Fig*-2.

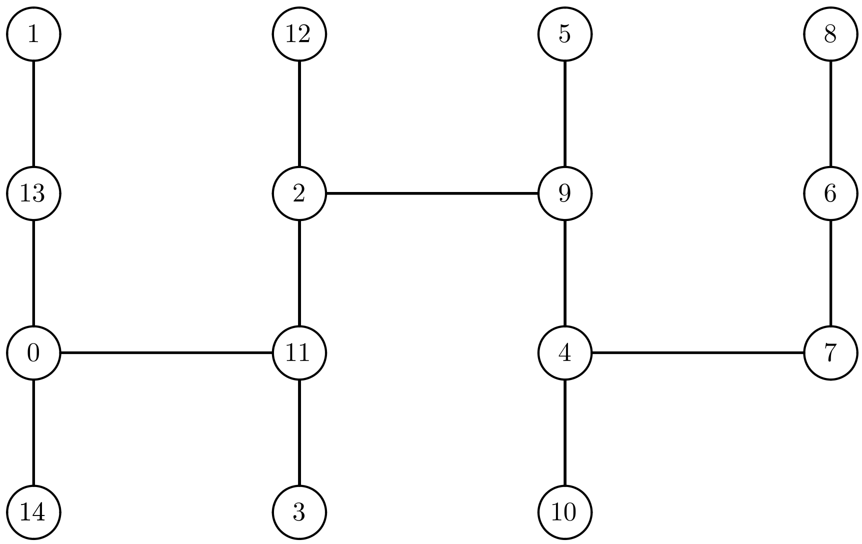

Fig-2: A graceful labeling of a T_P-tree using theorem 1.

REMARK. We shall show further that f is indeed a (k,d)-*balanced* labeling of T. Since i and j are of opposite parity, without loss of generality, we may assume that i is odd and j is even.

Case 1: n is even (i.e. q is even)

Since $i \leq n+1$, we get

$$\begin{aligned} f(v_i) &= k+(q-1)d-((i-1)/2)d \\ &\geq k+(q-1)d-((q+1-1)/2)d \\ &= k+((q/2)-1)d \\ &> ((q/2)-1)d \\ &= \lceil (q-1)/2 \rceil d \end{aligned}$$

where $\lceil . \rceil$ denote the greatest integer functions. The second last inequality holds since $k \geq 1$ and the last equality holds since q is even. Also,

$$f(v_j) = ((j/2)-1)d \leq ((q/2)-1)d = \lceil (q-1)/2 \rceil d.$$

Thus we get

$$f(v_j) \leq \lceil (q-1)/2 \rceil d < f(v_i).$$

Case 2: n is odd

By means of aruguments similar to those in Case 1,

$$f(v_j) \leq \lceil (q-1)/2 \rceil d < f(v_i).$$

As i and j are arbitrarily chosen so that v_iv_j is an edge in T, it follows that f is also a (k,d)-*balanced* labeling of T with characteristic $\lceil (q-1)/2 \rceil d$.

THEOREM 2. If T is a T_P-tree with q edges then the subdivision tree $S(T)$ is (k,d)-*graceful* for all positive integers k and d.

PROOF. Let T be a T_P-tree with n vertices and q edges. By the definition of a T_P-tree there exists a parallel transformation P of T so that we get $P(T)$. Denote the vertices of $P(T)$ successively as $v_1, v_2, ..., v_n$ starting from one pendant vertex of $P(T)$ right up to other and preserve the same for T.

Construct the subdivision tree $S(T)$ of T by introducing exactly one vertex between every edge v_iv_j with $i < j$ of T and denote the vertex as $v_{i,j}$. Let $v_{m^x}v_{h^x}$, $x = 1, 2, ..., z$ be the z transformable edges of T with $m^x < m^x + 1$ for all x. Let t_x be the path length from the vertex v_{m^x} to the corresponding pendant vertex decided by the transformable edge $v_{m^x}v_{h^x}$ of T.

Define a labeling $f : V(S(T)) \to \{0, 1, 2, \ldots, k + (2q-1)d\}$ by $f(v_i) = k + (2q-1)d - (i-1)d$ for $i = 1, 2, ..., n$ and

$$\begin{array}{ll} f(v_{i,j}) = (i-1)d, & j \neq i+1 \\ f(v_{i,j}) = id, & j = i+1; i = m^c, m^c+1, ..., m^c+t_c-1; c = 1, 2, ..., z, \\ f(v_{i,j}) = (i-1)d, & j = i+1; i \neq m^c, m^c+1, ..., m^c+t_c-1; c = 1, 2, ..., z, \end{array}$$

where k and d are positive integers and $2q$ is the number of edges of $S(T)$.

Let

$$A = \{v : v \in V(S(T)) \text{ with } v = v_i, i = 1, 2, ..., n\}$$

and

$$B = \{v : v \in V(S(T)) \text{ with } v = v_{i,j}, i = 1, 2, ..., n-1; j = 2, 3, ..., n\}.$$

Then by the definition of f above, the least value $k+(q-1)d$ on the set $f(A)$ is greater than the greatest value $(q-1)d$ on the set $f(B)$. Clearly f is injective from A to $f(A)$. Also f assigns values to the members $v_{i,j}$ of B with $j = i+1, i = 1, 2, ..., n-1$, in strictly increasing order and the increasing order gets uniformity due to values on the members $v_{i,j}$ of B with $j \neq i+1$. Therefore f is injective.

Now by the definition of induced edge function g_f for graceful labeling f, we get, the greatest and least values on the edges as follows:

$$g_f(v_1v_{1,2}) = |f(v_1) - f(v_{1,2})| = k + (2q-1)d$$

and

$$\begin{aligned} g_f(v_{n-1,n}v_n) &= g_f(v_{q,q+1}v_{q+1}) \\ &= |f(v_{q,q+1}) - f(v_{q+1})| \\ &= |(q-1)d - k - (q-1)d| = k. \end{aligned}$$

As we have uniform increasing order of values on the vertices due to f and there are $2q$ edges in $S(T)$, clearly g_f is injective with edge values forms the set $\{k, k+d, ..., k+(2q-1)d\}$. Hence f is a (k,d)-*graceful* labeling of $S(T)$. The proof is complete.

For example, a $(1,1)$-*graceful* labeling of subdivision of a T_P-tree using theorem 2, is shown in *Fig*-3.

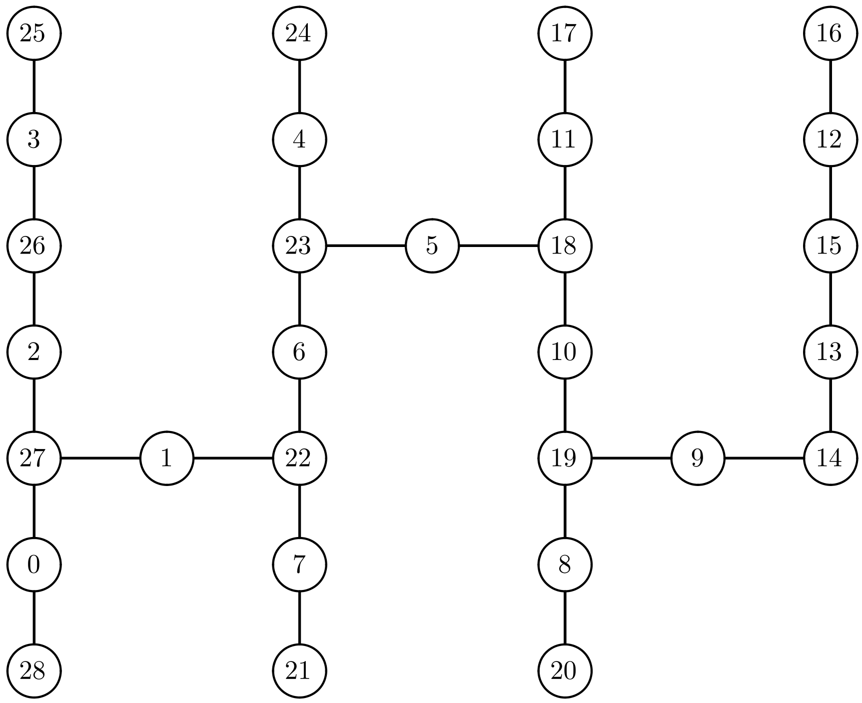

Fig-3: A graceful labeling of subdivision of a T_P-tree using theorem 2.

References

[1] B. D. Acharya, Personal communication.

[2] B. D. Acharya and S. M. Hegde, Arithmetic graphs, J. Graph Theory, 14(3)(1990), 275–299.

[3] G. S. Bloom and S. W. Golomb, Applications of numbered undirected graphs, Proceedings of the IEE, 165(4)(1977), 562–570.

[4] J. A. Gallian, A dynamic survey of graph labeling, The Electronic journal of combinatorics, 5(1)(1998), Dynamic Survey 6, 43 pp.

[5] F. Harary, Graph Theory, Addison-Wesley, Reading, Massachusetts, 1972.

[6] A. Rosa, On certain valuations of the vertices of a graph, Theory of graphs (Internat. Symp, Rome, July 1966), Gordan and Breach, N.Y and Paris, 1967, 349–355.