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Abstract. In this paper we consider regularized modified Newton's method for approximately solving the nonlinear ill-posed problem $F(x) = y$, where the right hand side is replaced by noisy data $y^\delta \in Y$ with $\|y - y^\delta\| \leq \delta$ and $F : D(F) \subset X \to Y$ is a nonlinear operator between Hilbert spaces X and Y. Under the assumption that Fréchet derivative F' of F is Lipschitz continuous, a choice of the regularization parameter and a stopping rule based on a majorizing sequence are presented. We prove that under a general source condition on $x_0 - \hat{x}$, the error $\|\hat{x} - x^\delta_{k,\alpha}\|$ between the regularized approximation $x^\delta_{k,\alpha}$ ($x_0 := x^\delta_{0,\alpha}$) and the solution $\hat{x}$ is of optimal order.

Keywords. Tihkonov regularization, regularized Newton's method, balancing principle.

2000 Mathematics Subject Classification. 65J20, 65J15, 47J06.

1 Introduction

In this paper we consider nonlinear ill-posed problems

$$F(x) = y, \tag{1.1}$$

where $F : D(F) \subset X \to Y$ is a nonlinear operator with non-closed range $R(F)$ and X, Y are infinite dimensional real Hilbert spaces with corresponding inner product $\langle ., . \rangle$ and norm $\|.\|$ respectively. We assume throughout that

- Equation (1.1) has a solution $\hat{x}$ (not necessarily unique), which in general does not depend continuously on the right hand side data y.
- only noisy data $y^\delta \in Y$ with

$$\|y - y^\delta\| \leq \delta \tag{1.2}$$

  are available.
- F possesses a locally uniformly bounded Fréchet derivative $F'(.)$ in a ball $B_r(\hat{x})$ of radius r around $\hat{x} \in X$.

Since (1.1) is ill-posed, one has to regularize (1.1). Tikhonov regularization (cf. [6–10, 20]) is one of the most widely used regularization method for solving linear and nonlinear ill-posed problems. In this method a regularized approximation x_α^δ is obtained by solving the minimization problem

$$\min_{x \in D(F)} J_\alpha(x), \quad J_\alpha(x) = \|F(x) - y^\delta\|^2 + \alpha\|x - x_0\|^2 \tag{1.3}$$

with an initial guess $x_0 \in X$ and a properly chosen regularization parameter $\alpha > 0$. It is known [21]that if x_α^δ is an interior point of $D(F)$, then the regularized approximation x_α^δ satisfies the Euler equation

$$F'(x)^*(F(x) - y^\delta) + \alpha(x - x_0) = 0 \tag{1.4}$$

of Tikhonov functional $J_\alpha(x)$. Here and below $F'(x)^*$ is the adjoint of the Fréchet derivative $F'(x)$.

Convergence of a global minimizer of (1.3) was established in [7, 19, 21] and convergence rates under logarithmic source conditions on $x_0 - \hat{x}$ have been established in [15].

Many authors considered iterative methods like Landweber's method [4, 5, 11], iteratively regularized Gauss–Newtons method [3, 12, 13], etc. for solving (1.1). In [15] fixed point iteration

$$x^0 = \mathrm{Proj}_{D(F)} x_0, x^{k+1} = \mathrm{Proj}_{D(F)} \Phi(x^k) \tag{1.5}$$

with $\Phi(x) = x - (F'(x)^* F'(x) + \alpha I)^{-1}[F'(x)^*(F(x) - y^\delta) + \alpha(x - x_0)]$ where $\mathrm{Proj}_{D(F)}$ is the projection (with respect to the norm on X) on $D(F)$ has been considered and proved that x^k converges to the stationary point x_α^δ of the Tikhonov functional $J_\alpha(x)$. However no error estimate for $\|x^k - \hat{x}\|$ has been given in [15]. In [16], a general sequence (z_l^α) converging to the solution x_α^δ of the Tikhonov functional $J_\alpha(x)$ were considered and obtained estimate for $\|z_l^\alpha - \hat{x}\|$, under the following assumptions.

Assumption 1.1. There exists $v \in X$ such that

$$x_0 - \hat{x} = \varphi(F'(\hat{x})^* F'(\hat{x}))v \tag{1.6}$$

where $\varphi : [0, \sigma] \to R^+$, $\sigma > \|F'(\hat{x})\|^2$, and the function $\Psi(\lambda) = \frac{\varphi(\lambda)}{\lambda^{1/2}}$ is non-decreasing. Moreover φ satisfies,

$$æ\frac{\alpha}{\varphi(\alpha)} \leq \inf_{\alpha \leq \lambda \leq \sigma} \frac{\lambda}{\varphi(\lambda)}, \quad 0 < \alpha \leq \sigma.$$

In this paper we consider a modified form called regularized modified Newton's method defined iteratively by

$$x_{n+1,\alpha}^{\delta} = x_{n,\alpha}^{\delta} - (F'(x_0)^* F'(x_0) + \alpha I)^{-1}[F'(x_0)^*(F(x_{n,\alpha}^{\delta}) - y^{\delta}) + \alpha(x_{n,\alpha}^{\delta} - x_0)], \tag{1.7}$$

$x_{0,\alpha}^{\delta} := x_0$ for solving (1.1). Note that the methods considered in [14, 19, 21], require the computation of the Fréchet derivative $F'(.)$ at global minimizers x_{α}^{δ} of the Tikhonov functional $J_\alpha(x)$ and the methods considered in [15] require the computation of the Fréchet derivative $F'(.)$ at each iteration $x_{n,\alpha}^{\delta}$ converging to the global minimizer of the Tikhonov functional $J_\alpha(x)$.

Observe that the method (1.7) requires the computation of Fréchet derivative $F'(.)$ only at one point x_0. This is one of the advantage of the method considered in this paper. Another advantage of our method is that the stopping rule in this paper is based on a majorizing sequence (see [1]) and hence it is not depending on the method.

In Section 2 we provide some preparatory result and derive error bounds for $\|x_{\alpha}^{\delta} - \hat{x}\|$ under certain general source conditions which include the logarithmic source conditions consider in [15]. In Section 3 we consider the regularized modified Newton's method defined in (1.7) and prove that $x_{n,\alpha}^{\delta}$ converges to the solution x_{α}^{δ} of (1.2). The analysis of this section is based on a majorizing sequence. Also in this section, using an error estimate for $\|x_{\alpha}^{\delta} - \hat{x}\|$, we obtain an error estimate for $\|x_{n,\alpha}^{\delta} - \hat{x}\|$. In Section 4 we derive error bounds for $\|x_{n,\alpha}^{\delta} - \hat{x}\|$ by choosing the regularization parameter α by an a priori as well as by the balancing principle proposed by Pereverzev and Schock [18]. Algorithm for implementing the balancing principle and the stopping rule for the iteration are given in Section 5 and finally the paper ends with some concluding remarks in Section 6.

2 Preparatory results

Throughout this paper we assume that the operator F satisfies the following assumptions.

Assumption 2.1. There exists $r > 0$ such that $B_r(\hat{x}) \subseteq D(F)$ and F is Fréchet differentiable at all $x \in B_r(\hat{x})$.

Assumption 2.2. There exists a constant $k_0 > 0$ such that for every $x, u \in B_r(\hat{x})$ and $v \in X$, there exists an element $\Phi(x, u, v) \in X$ satisfying

$$[F'(x) - F'(u)]v = F'(u)\Phi(x, u, v), \|\Phi(x, u, v)\| \leq k_0 \|v\| \|x - u\|$$

for all $x, u \in B_r(\hat{x})$ and $v \in X$.

The next assumption on source condition is based on a source function φ and a property of the source function φ. We will use this assumption to obtain an error estimate for $\|\hat{x} - x_\alpha^\delta\|$.

Assumption 2.3. There exists a continuous, strictly monotonically increasing function $\varphi : (0, a] \to (0, \infty)$ with $a \geq \|F'(\hat{x})^* F'(\hat{x})\|$ satisfying

- $\lim_{\lambda \to 0} \varphi(\lambda) = 0$
- for $\alpha \leq 1$, $\varphi(\alpha) \geq \alpha$
- $\sup_{\lambda \geq 0} \frac{\alpha \varphi(\lambda)}{\lambda + \alpha} \leq c_\varphi \varphi(\alpha), \quad \forall \lambda \in (0, a]$
- there exists $v \in X$ such that

$$x_0 - \hat{x} = \varphi(F'(\hat{x})^* F'(\hat{x}))v. \tag{2.1}$$

Remark 2.4. The above asumption on source function φ includes the logarithmic source condition considered in [15].

We will be using the following theorems from [21] for our error analysis.

Theorem 2.5 (cf. [21], Theorem 2.7). *Let x_α^δ be the solution of the regularized problem* (1.4) *and x_α be a solution of* (1.4) *with y^δ replaced by the exact data y. Assume Assumption 2.2 with radius $r = \frac{\delta}{\sqrt{\alpha}} + 2\|x_0 - \hat{x}\|$. If $k_0\|x_0 - \hat{x}\| < 1$, then*

$$\|x_\alpha^\delta - x_\alpha\| \leq \frac{\delta}{\sqrt{\alpha}\sqrt{1 - k_0\|x_0 - \hat{x}\|}}. \tag{2.2}$$

The following theorem is essentially a reformulation of Theorem 2.6 proved in [21]. For the sake of completion, we supply its proof as well.

Theorem 2.6. *Let x_α be a solution* (1.4) *with y^δ replaced by the exact data y. Assume Assumption 2.2 and Assumption 2.3 with radius $r = 2\|x_0 - \hat{x}\|$. If $2k_0\|x_0 - \hat{x}\| < 1$, then*

$$\|x_\alpha - \hat{x}\| \leq \frac{c_\varphi \varphi(\alpha)\|v\|}{\sqrt{1 - 2k_0\|x_0 - \hat{x}\|}}. \tag{2.3}$$

Proof. Note that $F'(x_\alpha)^*(F(x_\alpha)-y)+\alpha(x_\alpha-x_0)=0$, so

$$\begin{aligned}
&(F'(\hat{x})^*F'(\hat{x})+\alpha I)(x_\alpha-\hat{x})\\
&\quad=(F'(\hat{x})^*F'(\hat{x})+\alpha I)(x_\alpha-\hat{x})-F'(x_\alpha)^*(F(x_\alpha)-y)-\alpha(x_\alpha-x_0)\\
&\quad=\alpha(x_0-\hat{x})+F'(\hat{x})^*F'(\hat{x})(x_\alpha-\hat{x})-F'(x_\alpha)^*(F(x_\alpha)-y)\\
&\quad=\alpha(x_0-\hat{x})+F'(\hat{x})^*[F'(\hat{x})(x_\alpha-\hat{x})-(F(x_\alpha)-y)]\\
&\qquad-[F'(x_\alpha)^*-F'(\hat{x})^*](F(x_\alpha)-y)\\
&\quad=\alpha(x_0-\hat{x})+F'(\hat{x})^*[F'(\hat{x})(x_\alpha-\hat{x})-(F(x_\alpha)-F(\hat{x}))]\\
&\qquad-[F'(x_\alpha)^*-F'(\hat{x})^*](F(x_\alpha)-F(\hat{x})).
\end{aligned}$$

Thus

$$x_\alpha-\hat{x}=s_1+s_2+s_3 \tag{2.4}$$

where

$$\begin{aligned}
s_1&:=\alpha(F'(\hat{x})^*F'(\hat{x})+\alpha I)^{-1}(x_0-\hat{x}),\\
s_2&:=-(F'(\hat{x})^*F'(\hat{x})+\alpha I)^{-1}[F'(x_\alpha)^*-F'(\hat{x})^*](F(x_\alpha)-F(\hat{x})),
\end{aligned}$$

and

$$s_3:=(F'(\hat{x})^*F'(\hat{x})+\alpha I)^{-1}F'(\hat{x})^*[F'(\hat{x})(x_\alpha-\hat{x})-(F(x_\alpha)-F(\hat{x}))].$$

It follows from estimates (2.15) and (2.17) in [21] (also see estimate (A.7) in [14]), that

$$\|s_2\|\le k_0\|x_0-\hat{x}\|\|x_\alpha-\hat{x}\|$$

and

$$\|s_3\|\le k_0\|x_0-\hat{x}\|\|x_\alpha-\hat{x}\|.$$

Therefore by (2.4), to complete the proof it is enough to prove that $\|s_1\|\le c_\varphi\varphi(\alpha)\|v\|$. But by Assumption 2.3, we have

$$\begin{aligned}
\|s_1\|&=\|\alpha(F'(\hat{x})^*F'(\hat{x})+\alpha I)^{-1}\varphi(F'(\hat{x})^*F'(\hat{x}))v\|\\
&\le c_\varphi\varphi(\alpha)\|v\|.
\end{aligned}$$

This completes the proof of the theorem. □

3 Regularized modified newton's method

In this section we prove that $x_{n,\alpha}^{\delta}$ converges to x_{α}^{δ} and provide an error estimate for $\|x_{n,\alpha}^{\delta} - x_{\alpha}^{\delta}\|$. Our analysis is based on a majorizing sequence. Recall (see [1], Definition 1.3.11) that a nonnegative sequence (t_n) is said to be a majorizing sequence of a sequence (x_n) in X if

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n, \quad \forall n \geq 0.$$

We use the sequence $(t_n), n \geq 0$, given by $t_0 = 0, t_1 = \eta$,

$$t_{n+1} = t_n + \frac{k_0 \eta}{(1-\tilde{r})}(t_n - t_{n-1}) \tag{3.1}$$

where $\tilde{r} \in [0, 1)$, as a majorizing sequence, of the sequence $(x_{n,\alpha}^{\delta})$. The following lemma is essentially a reformulation of a lemma in [8].

Lemma 3.1. *Assume there exist nonnegative numbers k_0, η and $\tilde{r} \in [0,1)$ such that*

$$\frac{k_0}{(1-\tilde{r})}\eta \leq \tilde{r}. \tag{3.2}$$

Then the sequence (t_n) defined in (3.1) is increasing, bounded above by $t^{**} := \frac{\eta}{1-\tilde{r}}$, and converges to some t^* such that $0 < t^* \leq \frac{\eta}{1-\tilde{r}}$. Moreover, for $n \geq 0$,

$$0 \leq t_{n+1} - t_n \leq \tilde{r}(t_n - t_{n-1}) \leq \tilde{r}^n \eta, \tag{3.3}$$

and

$$t^* - t_n \leq \frac{\tilde{r}^n}{1-\tilde{r}}\eta. \tag{3.4}$$

Proof. Since the result holds for $\eta = 0, k_0 = 0$ or $r = 0$, we assume that $k_0 \neq 0, \eta \neq 0$ and $\tilde{r} \neq 0$. Observe that $t_1 - t_0 = \eta \geq 0$, assume that $t_{i+1} - t_i \geq 0$, for all $i \leq k$ for some k. Then $t_{k+2} - t_{k+1} = \frac{\kappa_0 \eta}{(1-\tilde{r})}(t_{k+1} - t_k) \geq 0$, so by induction $t_{n+1} - t_n \geq 0$ for all $n \geq 0$. Now since

$$\frac{k_0 \eta}{(1-\tilde{r})} \leq \tilde{r}$$

the estimate (3.3) follows from (3.1). Further observe that

$$\begin{aligned} t_{k+1} \leq t_k + \tilde{r}(t_k - t_{k-1}) &\leq \cdots \leq \eta + \tilde{r}\eta + \cdots + \tilde{r}^k \eta \\ &= \frac{1-\tilde{r}^{k+1}}{1-\tilde{r}}\eta < \frac{\eta}{1-\tilde{r}}. \end{aligned}$$

Hence the sequence $(t_n), n \geq 0$ is bounded above by $\frac{\eta}{1-\tilde{r}}$; nondecreasing, so it converges to some $t^* \leq \frac{\eta}{1-\tilde{r}}$, and

$$t^* - t_n = \lim_{i\to\infty} t_{n+i} - t_n \leq \lim_{i\to\infty} \sum_{j=0}^{i-1} (t_{n+1+j} - t_{n+j}) \leq \frac{\tilde{r}^n}{1-\tilde{r}}\eta.$$

That completes the proof of the lemma. □

To prove the convergence of the sequence $(x_{n,\alpha}^{\delta})$ defined in (1.7) we introduce the following notations: Let $R_\alpha(x_0) := F'(x_0)^* F'(x_0) + \alpha I$ and

$$G(x) := x - R_\alpha(x_0)^{-1}[F'(x_0)^*(F(x) - y^\delta) + \alpha(x - x_0)]. \tag{3.5}$$

Note that with the above notation, $G(x_{n,\alpha}^{\delta}) = x_{n+1,\alpha}^{\delta}$ and

$$\|R_\alpha(x_0)^{-1} F'(x_0)^* F'(x_0)\| \leq 1. \tag{3.6}$$

The following lemma based on the Assumption 2.2 will be used in due course.

Lemma 3.2. *For $u, v \in B_r(x_0)$*

$$F(v) - F(u) - F'(x_0)(v-u) = F'(x_0)\int_0^1 \Phi(u + t(v-u), x_0, v-u)dt.$$

Proof. Using the fundamental theorem of integration, for $u, v \in B_r(x_0)$ we have

$$F(v) - F(u) = \int_0^1 F'(u + t(v-u))(v-u)dt$$

so by Assumption 2.2 we have

$$F(v) - F(u) - F'(x_0)(v-u) = F'(x_0)\int_0^1 \Phi(u + t(v-u), x_0, v-u)dt.$$

This completes the proof of the lemma. □

Hereafter we assume that

$$r \geq \max\left\{\frac{\delta}{\sqrt{\alpha}} + 2\|x_0 - \hat{x}\|, t^{**}\right\}.$$

Theorem 3.3. *Let the assumptions in Lemma 3.1 with* $\eta = \frac{\|F(x_0)-y^\delta\|}{\sqrt{\alpha}}$ *and Assumption 2.2 be satisfied. Then the sequence* $(x_{n,\alpha}^\delta)$ *defined in* (1.7) *is well defined and* $x_{n,\alpha}^\delta \in B_{t^*}(x_0)$ *for all* $n \geq 0$*. Further* $(x_{n,\alpha}^\delta)$ *is Cauchy sequence in* $B_{t^*}(x_0)$ *and hence converges to* $x_\alpha^\delta \in \overline{B_{t^*}(x_0)} \subset B_{t^{**}}(x_0)$ *and* $F'(x_0)^*(F(x_\alpha^\delta) - y^\delta) + \alpha(x_\alpha^\delta - x_0) = 0$.

Moreover, the following estimates hold for all $n \geq 0$,

$$\|x_{n+1,\alpha}^\delta - x_{n,\alpha}^\delta\| \leq t_{n+1} - t_n, \tag{3.7}$$

and

$$\|x_{n,\alpha}^\delta - x_\alpha^\delta\| \leq \frac{\tilde{r}^n \|F(x_0) - y^\delta\|}{\sqrt{\alpha}(1-\tilde{r})}. \tag{3.8}$$

Proof. Let G be as in (3.5). Then for $u, v \in B_{t^*}(x_0)$,

$$\begin{aligned} G(u) - G(v) = u - v &- R_\alpha(x_0)^{-1}[F'(x_0)^*(F(u) - y^\delta) + \alpha(u - x_0)] \\ &+ R_\alpha(x_0)^{-1}[F'(x_0)^*(F(v) - y^\delta) + \alpha(v - x_0)] \\ &= R_\alpha(x_0)^{-1}[R_\alpha(x_0)(u - v) - F'(x_0)^*(F(u) - F(v))] \\ &+ \alpha R_\alpha(x_0)^{-1}(v - u) \\ &= R_\alpha(x_0)^{-1}F'(x_0)^*[F'(x_0)(u - v) - (F(u) - F(v)) + \alpha(u - v)] \\ &+ \alpha R_\alpha(x_0)^{-1}(v - u) \\ &= R_\alpha(x_0)^{-1}F'(x_0)^*[F'(x_0)(u - v) - (F(u) - F(v))] \end{aligned}$$

Thus by Lemma 3.2, Assumption 2.2 and (3.6) we have

$$\|G(u) - G(v)\| \leq k_0 t^* \|u - v\|. \tag{3.9}$$

Now we shall prove that the sequence (t_n) where (t_n) defined in Lemma 3.1 is a majorizing sequence of the sequence $(x_{n,\alpha}^\delta)$ and $x_{n,\alpha}^\delta \in B_{t^*}(x_0)$, for all $n \geq 0$.

Note that $\|x_{1,\alpha}^\delta - x_0\| = \|R_\alpha(x_0)^{-1}F'(x_0)^*(F(x_0) - y^\delta)\| \leq \frac{\|F(x_0)-y^\delta\|}{\sqrt{\alpha}} = \eta = t_1 - t_0$, assume that

$$\|x_{i+1,\alpha}^\delta - x_{i,\alpha}^\delta\| \leq t_{i+1} - t_i, \quad \forall i \leq k \tag{3.10}$$

for some k. Then

$$\begin{aligned} \|x_{k+1,\alpha}^\delta - x_0\| &\leq \|x_{k+1,\alpha}^\delta - x_{k,\alpha}^\delta\| + \|x_{k,\alpha}^\delta - x_{k-1,\alpha}^\delta\| + \cdots + \|x_{1,\alpha}^\delta - x_0\| \\ &\leq t_{k+1} - t_k + t_k - t_{k-1} + \cdots + t_1 - t_0 \\ &= t_{k+1} \leq t^*. \end{aligned}$$

So $x^{\delta}_{i+1,\alpha} \in B_{t^*}(x_0)$ for all $i \le k$, and hence, by (3.9) and (3.10),

$$\|x^{\delta}_{k+2,\alpha} - x^{\delta}_{k+1,\alpha}\| \le k_0 t^* \|x^{\delta}_{k+1,\alpha} - x^{\delta}_{k,\alpha}\| \le \frac{k_0 \eta}{(1-\tilde{r})}(t_{k+1} - t_k) = t_{k+2} - t_{k+1}.$$

Thus by induction $\|x^{\delta}_{n+1,\alpha} - x^{\delta}_{n,\alpha}\| \le t_{n+1} - t_n$ for all $n \ge 0$ and hence $(t_n), n \ge 0$ is a majorizing sequence of the sequence $(x^{\delta}_{n,\alpha})$. In particular $\|x^{\delta}_{n,\alpha} - x_0\| \le t_n \le t^*$, i.e., $x^{\delta}_{n,\alpha} \in B_{t^*}(x_0)$, for all $n \ge 0$. So $(x^{\delta}_{n,\alpha}), n \ge 0$ is a Cauchy sequence and converges to some $x^{\delta}_{\alpha} \in \overline{B_{t^*}(x_0)} \subset B_{t^{**}}(x_0)$ and

$$\|x^{\delta}_{\alpha} - x^{\delta}_{n,\alpha}\| \le t^* - t_n \le \frac{\tilde{r}^n \eta}{(1-\tilde{r})} = \frac{\tilde{r}^n \|F(x_0) - y^{\delta}\|}{\sqrt{\alpha}(1-\tilde{r})}.$$

Now by letting $n \to \infty$ in (1.7) we obtain $F'(x_0)^*(F(x^{\delta}_{\alpha}) - y^{\delta}) + \alpha(x^{\delta}_{\alpha} - x_0) = 0$. This completes the proof of the theorem. □

4 Error bounds

Combining the estimates in Theorem 2.5, Theorem 2.6 and Theorem 3.3 we obtain the following,

Theorem 4.1. *Let* $x^{\delta}_{n,\alpha}$ *be as in* (1.7) *and let the assumptions in Theorem 2.5, Theorem 2.6 and Theorem 3.3 be satisfied. Then*

$$\|x^{\delta}_{n,\alpha} - \hat{x}\| \le \frac{\|F(x_0) - y^{\delta}\|}{(1-\tilde{r})} \frac{\tilde{r}^n}{\sqrt{\alpha}} + \frac{1}{\sqrt{1 - k_0\|x_0 - \hat{x}\|}} \frac{\delta}{\sqrt{\alpha}} + \frac{c_{\varphi}\|v\|}{\sqrt{1 - 2k_0\|x_0 - \hat{x}\|}} \varphi(\alpha). \tag{4.1}$$

Let

$$n_{\delta} := \min\{n : \tilde{r}^n \le \delta\} \tag{4.2}$$

and let

$$C := \max\left\{\left(\frac{\|F(x_0) - y^{\delta}\|}{1-\tilde{r}} + \frac{1}{\sqrt{1 - k_0\|x_0 - \hat{x}\|}}\right), \frac{c_{\varphi}\|v\|}{\sqrt{1 - 2k_0\|x_0 - \hat{x}\|}}\right\}. \tag{4.3}$$

Theorem 4.2. *Let* $x^{\delta}_{n,\alpha}$ *be as in* (1.7)*,* n_{δ} *be as in* (4.2) *and* C *be as in* (4.3)*. Let the assumptions in Theorem 4.1 be satisfied. Then*

$$\|x^{\delta}_{n_{\delta},\alpha} - \hat{x}\| \le C\left(\frac{\delta}{\sqrt{\alpha}} + \varphi(\alpha)\right) \tag{4.4}$$

4.1 A priori choice of the parameter

Note that, if the regularization parameter $\alpha_\delta := \alpha(\delta)$ satisfies

$$\sqrt{\alpha_\delta}\varphi(\alpha_\delta) = \delta, \tag{4.5}$$

then the error $\frac{\delta}{\sqrt{\alpha}} + \varphi(\alpha)$ is of optimal order.

Let $\chi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}, 0 < \lambda \leq a$. Then by (4.5), $\delta = \chi(\varphi(\alpha_\delta))$, so $\alpha_\delta = \varphi^{-1}(\chi^{-1}(\delta))$. Hence for the a priori choice $\alpha = \varphi^{-1}(\chi^{-1}(\delta))$ we have the following theorem.

Theorem 4.3. *Let $\chi(\lambda) = \lambda\sqrt{\varphi^{-1}(\lambda)}$ for $0 < \lambda \leq a$, and assumptions in Theorem 4.2 holds. For $\delta > 0$, let $\alpha = \varphi^{-1}(\chi^{-1}(\delta))$ and let n_δ be as in* (4.2). *Then*

$$\|x^\delta_{n_\delta,\alpha} - \hat{x}\| = O(\chi^{-1}(\delta)).$$

The disadvantage of the above a priori parameter choice is that, in practice an index function φ describing a solution smoothness is usually unknown. So one has to apply a posteriori rules which choose α from quantities that arise during calculations. The well-known a posteriori rules that have been used for Tikhonov regularization of nonlinear ill-posed problems are the discrepancy principle [2,22], the rules considered in [15, 21] and the balancing principle considered in [18]. In the next subsection we considere the balancing principle considered in [18] for choosing the regularization parameter α in (1.7).

4.2 An adaptive choice of the parameter

In this subsection, we will present the balancing principle studied in [16–18] for choosing the parameter α in (1.7).

In the balancing principle, the regularization parameter α is selected from some finite set

$$D_M(\alpha) := \{\alpha_i = \mu^i\alpha_0, i = 0, 1, \ldots, M\} \tag{4.6}$$

where $\mu > 1$ and M is such that $1 \leq \alpha_M$. We choose $\alpha_0 := \delta$, because we expect to have an accuracy of order at least $O(\sqrt{\delta})$ and from Theorem 4.2, it follows that such an accuracy cannot be guaranteed for $\alpha < \delta$.

Let $x_i := x^\delta_{n_\delta,\alpha_i}$. Then by (3.8) and (4.2) we have

$$\|x_i - x^\delta_{\alpha_i}\| \leq \frac{\|F(x_0) - y^\delta\|}{1 - \tilde{r}}\frac{\delta}{\sqrt{\alpha_i}}, \quad \forall i = 0, 1, \ldots, M. \tag{4.7}$$

The parameter choice strategy that we are going to consider in this paper, we selects $\alpha = \alpha_i$ from $D_M(\alpha)$ and operates only with corresponding x_i, $i = 0, 1, \ldots, M$.

Theorem 4.4. *Assume that there exists* $i \in \{0, 1, 2, \ldots, M\}$ *such that* $\varphi(\alpha_i) \le \frac{\delta}{\sqrt{\alpha_i}}$. *Let assumptions of Theorem 4.2 be satisfied and let*

$$l := \max\left\{i : \varphi(\alpha_i) \le \frac{\delta}{\sqrt{\alpha_i}}\right\} < M,$$
$$k := \max\left\{i : \|x_i - x_j\| \le 4C\frac{\delta}{\sqrt{\alpha_j}},\ j = 0, 1, 2, \ldots, i\right\} \tag{4.8}$$

where C *is as in* (4.3). *Then* $l \le k$ *and*

$$\|\hat{x} - x_k\| \le c\psi^{-1}(\delta)$$

where $c = 6C\sqrt{\mu}$.

Proof. To see that $l \le k$, it is enough to show that, for each $i \in \{1, 2, \ldots, M\}$,

$$\varphi(\alpha_i) \le \frac{\delta}{\sqrt{\alpha_i}} \Longrightarrow \|x_i - x_j\| \le 4C\frac{\delta}{\sqrt{\alpha_j}}, \quad \forall j = 0, 1, \ldots, i.$$

For $j \le i$, by (4.4) we have

$$\begin{aligned}\|x_i - x_j\| &\le \|x_i - x^\dagger\| + \|x^\dagger - x_j\| \\ &\le C\left(\varphi(\alpha_i) + \frac{\delta}{\sqrt{\alpha_i}}\right) + C\left(\varphi(\alpha_j) + \frac{\delta}{\sqrt{\alpha_j}}\right) \\ &\le 4C\frac{\delta}{\sqrt{\alpha_j}}.\end{aligned}$$

Thus the relation $l \le k$ is proved. Next we observe that

$$\begin{aligned}\|\hat{x} - x_k\| &\le \|\hat{x} - x_l\| + \|x_l - x_k\| \\ &\le C\left(\varphi(\alpha_l) + \frac{\delta}{\sqrt{\alpha_l}}\right) + 4C\frac{\delta}{\sqrt{\alpha_l}} \\ &\le 6C\frac{\delta}{\sqrt{\alpha_l}}.\end{aligned}$$

Now since $\alpha_\delta \le \alpha_{l+1} \le \mu\alpha_l$, it follows that

$$\frac{\delta}{\sqrt{\alpha_l}} \le \sqrt{\mu}\frac{\delta}{\sqrt{\alpha_\delta}} = \sqrt{\mu}\varphi(\alpha_\delta) = \sqrt{\mu}\psi^{-1}(\delta).$$

This completes the proof of the theorem. □

5 Implementation of adaptive choice rule

In this section we provide an algorithm for the determination of a parameter fulfilling the balancing principle (4.8) and also provide a starting point for the iteration (3.3) approximating the unique solution x_α^δ of (1.3). We choose the starting point x_0 such that $x_0 \in D(F)$ and $\|F(x_0) - y^\delta\| \leq \frac{\sqrt{\delta}}{4k_0}$.

The choice of the stopping index n_δ involves the following steps:

- Choose $\alpha_0 = \delta$ and $\mu > 1$.
- Choose $\tilde{r} > 0$ such that $\tilde{r} < \frac{1}{2}\big(1 - \sqrt{1 - \frac{4k_0\|F(x_0)-y^\delta\|}{\sqrt{\delta}}}\big)$.
- Choose the parameter $\alpha_M = \mu^M \alpha_0$ big enough but not too large.
- Choose n_δ such that $n_\delta = \min\{n : \tilde{r}^n \leq \delta\}$.

Finally the adaptive algorithm associated with the choice of the parameter specified in Theorem 4.4 involves the following steps:

5.1 Algorithm

- Set $i \leftarrow 0$
- solve $x_i := x_{n_\delta,\alpha_i}^\delta$ by using the iteration (3.3).
- If $\|x_i - x_j\| > 4C\frac{\sqrt{\delta}}{\sqrt{\mu^j}}$, $j \leq i$, then take $k = i - 1$.
- Set $i = i + 1$ and return to step 2.

6 Concluding remarks

In this paper we have considered a regularized modified Newton's method for obtaining approximate solutions for a nonlinear operator equation $F(x) = y$, when the available data is y^δ in place of the exact data y. The procedure involves finding the fixed point of the function

$$G(x) = x - (F'(x_0) * F'(x_0) + \alpha I)^{-1}[F'(x_0)^*(F(x) - y^\delta) + \alpha(x - x_0)]$$

in an iterative manner.

It should of course be mentioned that the method requires to compute the Fréchet derivative $F'(.)$ only at one point x_0 unlike the method considered in [15]. Also it should be noted that the proof for the convergence result and the stopping rule are based on a majorizing sequence, which is independent of the method. For choosing the regularization parameter α we made use of the balancing principle suggested in [18].

In a future work, it is envisaged to investigate the method considered in [15] to obtain quadratic convergence of the iterate to the solution x_α^δ of the Tikhonov functional $J_\alpha(x)$.

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