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INFLUENCE LINES FOR BENDING MOMENTS IN BEAMS ON ELASTIC FOUNDATIONS

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Abstract—A finite difference method assuming parabolic variation of contact pressure distribution is presented to obtain the influence lines for bending moments in beams on an elastic foundation. These influence lines can conveniently be used to find moments in beams on elastic foundations due to any type of loads. The computational procedure presented is simple. Accurate results are obtained with only 10 elements.

1. INTRODUCTION

The continuous beam–slab type of foundation system is generally analysed as a beam on elastic foundations. Such a structural system for foundations ensures flexural behaviour while avoiding punching failure.

In general, to analyse, soil–structure interaction problems, the following two soil models with a large area of practical utility are considered. The Winkler model or local deformation hypothesis, in which the soil is considered as an elastic spring field and continuum model, and is also considered as a homogeneous, isotropic, linearly deformable elastic half space.

Soil is rarely homogeneous, generally layered and anisotropic, and never perfectly elastic. The deformations are time dependant and partly irreversible. For such a complex material, numerical methods with good approximations provide equally reliable results as the rigorous classical methods and have the advantage of being easily programmable for the computers.

Of the many solutions of beams on elastic foundations, Hetenyi [1] provided the classical solution of a fourth-order governing differential equation for the beam of uniform section on Winkler medium. Barden [2] computed the contact pressure distribution based on displacement compatibility between the beam and soil assuming a stepped variation of contact pressure. Popov [3] presented a method of successive approximations of contact pressure distribution untill the convergence. The elastic line of the beam is determined using the moment-area method. Bowles [4] provided a finite difference method assuming a stepped variation of contact pressure and a finite element method combining a conventional beam element with descrete springs at the ends of the beam. Lee and Harrison [5] derived a stiffness matrix for a beam on an elastic foundation using a slope–deflection method. The finite element method for a beam on elastic foundations is provided by many authors. Three-dimensional finite elements were also used for the soil–structure interaction problems. These require large computational time and effort, hence the requirement for a simpler numerical method without loss of accuracy.

The finite difference method described in this paper is applicable for beams of varying sections (i.e. moment of inertia) and varying subgrade modulus. The method presented assumes a very realistic contact pressure distribution, namely parabolic distribution, while most of the numerical methods assume a stepped distribution.

2. COMPUTATIONAL PROCEDURE

The governing differential equation for the beam (Fig. 1) is

$$(\mathrm{d}^2y/\mathrm{d}x^2) = (M_n/EI) \quad (1)$$

The bending moment M_n, at any point n is produced by the loads together with the contact pressure. The contact pressure may be represented by equivalent concentrated reactions, R_n, at the nodal points.

Assuming a parabolic distribution of contact pressure, equivalent concentrated reactions (Fig. 2) are

$$R_1 = -(k_s h/24)(7y_1 - 6y_2 - y_3)$$

$$R_m = -(k_s h/24)(7y_m + 6y_{m-1} - y_{m-2})$$

at the ends and

$$R_n = -(k_s h/24)(2y_{n-1} + 20y_n + 2y_{n+1})$$

at any intermediate point, where the deflections, y in the upward direction are positive.

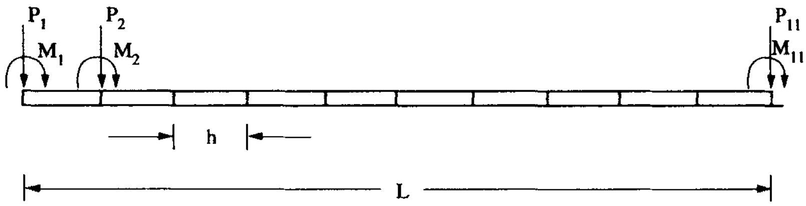


Fig. 1. Beam on elastic foundation with loading.

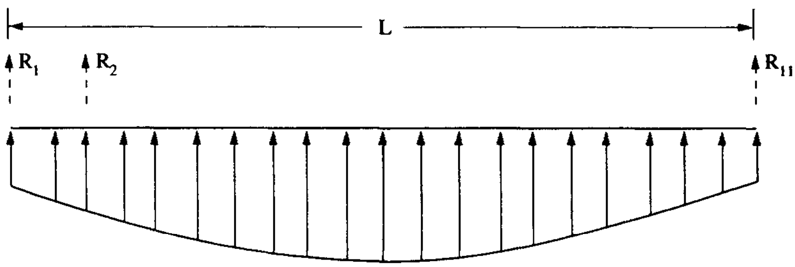


Fig. 2. Contact pressure distribution.

Writing the differential eqn (1) in the finite difference form,

$$y_{n-1} - 2y_n + y_{n+1} = (h^2/EI)M_n, \tag{2}$$

e.g. at node 2,

$$y_1 = 2y_2 + y_3 = (h^2/EI)[(R_1 h) - (P_1 h) + (M_1 + M_2)]$$

$$= (-k_s h^4/24EI)[7y_1 + 6y_2 - y_3]$$

$$+ (h^2/EI)[(-P_1 h) + (M_1 + M_2)].$$

Similar equations are written up to the $(m-1)$th node, and the same may be written in the matrix form as

$$[CY][y] = -\beta^4[CM][y] + (h^2/EI)([EMP] + [EMM])$$

$$[CY] + \beta^4[CM][Y] = (h^2/EI)[EM]$$

where

$$\beta = 4\sqrt{(k_s h^4/24EI}$$

$$[CYM][Y] = (h^2/EI)[EM]. \tag{3}$$

Thus eqn (3) is a matrix of $(m-2)$ equations. The remaining two equations are obtained from equilibrium. Hence m equations to solve for m deflections.

Summing up the vertical forces, i.e.

$$(R_1 + R_2 + \cdots R_m) - (P_1 + P_2 + \cdots P_m) = 0$$

and summing up the moments of forces about the last node, i.e.

$$[R_1(m-1)h + R_2(m-2)h + \cdots + R_{m-1}h]$$

$$-[P_1(m-1)h + P_2(m-2)h + \cdots + P_{m-1}h]$$

$$+[M_1 + M_2 + \cdots + M_m] = 0.$$

The equilibrium equations are also written in the matrix form as

$$[CYM][y] = (h^2/EI)[EM]$$

and appended to eqn (3).

The set of equations (3) are solved, to obtained the deflections as

$$[y] = [CYM]^{-1}[EM](h^2/EI)$$

$$= [y_0](h^2/EI) \tag{4}$$

and the reactions

$$[R] = (-k_s h/24)[CM][y_0](h^2/EI)$$

$$= -(\beta^4/h)[CM][y_0]. \tag{5}$$

The moments are obtained from

$$[M] = (-k_s h^2/24)[CM][y] + [EM]$$

$$= -\beta^4[CM][y_0] + [EM] \tag{6}$$

Influence lines for bending moments are obtained for different values of β (a non-dimensional parameter, combining modulus of subgrade reaction and rigidity of foundation), considering the unit length of the beam and unit loads and moments at the nodal points. These influence lines can be used to find the moments at the nodal points due to any type of load at any point (not necessarily at the nodes).

The above procedure can be conveniently used for beams of varying cross-sections and varying subgrade modulus by imposing appropriate β values at nodal points in eqn (3).

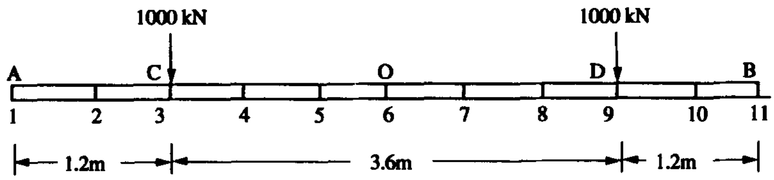
It is noticed that the moments obtained with only 10 elements are in good agreement with Hetenyi's classical solution.

3. NUMERICAL EXAMPLES

A computer program is written to get the influence coefficients for the moments, for unit load (or moment) and for unit length dividing the beam into 10 elements. The following examples are considered to explain the use of influence lines and to compare Hetenyi's solution [1] and the solution using influence lines.

3.1. *Solution to example* 1 (*Fig.* 3)

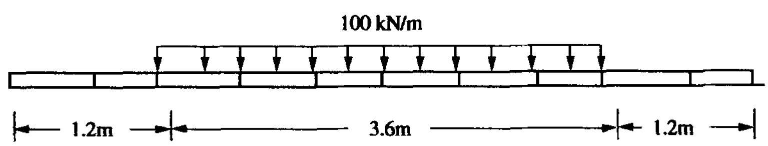
Non-dimensional parameter $(\beta) = 4\sqrt{(k_s h^4/24EI)} = 0.2$ from the influence lines for $\beta = 0.2$ (Fig. 9).



Example 1. Data for the beam, L = 6m, b = 1.5m, t = 0.5m
$E = 22 \times 10^6$ k N/m^2, h = (L/10) = 0.6m
For the soil, ks = 100 10^3 k N/m^2

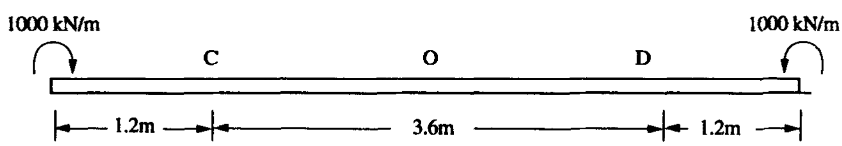
Example 2. Data for the beam and loading is same but ks = 500 10^3 kN/m^2

Fig. 3. Examples 1 and 2.



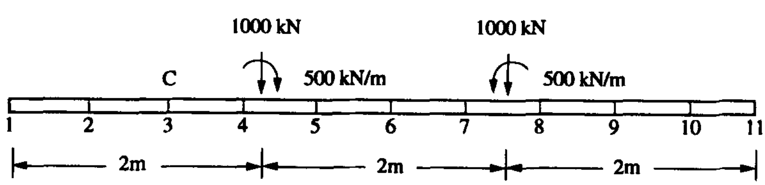
Example 3. Data for the beam and soil is same as example 1.

Fig. 4. Example 3.



Example 4. Data for the beam and soil is same as example 1.

Fig. 5. Example 4.



Example 5. Data for the beam and soil is same as example 1.

Fig. 6. Example 5.

Moment at C

$$=[(Im)_{3,3} \times P_3 \times L] + [(Im)_{3,9} \times P_9 \times L]$$

$$=(0.0512 \times 1000 \times 6) + (-0.0077 \times 1000 \times 6)$$

$$=261 \text{ kN m}.$$

Moment at 0

$$=[(Im)_{6,3} \times P_3 \times L] + [(Im)_{6,9} \times P_9 \times L]$$

$$=[-0.0188 \times 1000 \times 6] + [-0.0188 \times 1000 \times 6]$$

$$=-225.6 \text{ kN m}.$$

3.2. *Solution to example* 2 (*Fig.* 3)

Non-dimensional parameter, $\beta = 0.3$. From the influence line diagrams for $\beta = 0.3$ (Fig. 10), moment at C

$$=[(Im)_{3,3} \times P_3 \times L] + [(Im)_{3,9} \times P_9 \times L]$$

$$=[0.0466 \times 1000 \times 6] + [-0.00314 \times 1000 \times 6]$$

$$=260.76 \text{ kN m}.$$

Moment at 0

$$=[(Im)_{6,3} \times P_3 \times L] + [(Im)_{6,9} \times P_9 \times L]$$

$$=[-0.0125 \times 1000 \times 6] + [-0.0125 \times 1000 \times 6]$$

$$=-150.0 \text{ kN m}.$$

3.3. *Solution to example* 3 (*Fig.* 4)

From the influence lines for $\beta = 0.2$ (Fig. 9), moment at 0

$$=(\text{Area† under influence line } (Im)_{6,j})$$

$$\times \text{Intensity of load} \times L$$

$$=(h/2)[(Im)_{6,3} + 2((Im)_{6,4} + (Im)_{6,5} + (Im)_{6,6}$$

$$+(Im)_{6,7} + (Im)_{6,8}) + (Im)_{6,9}] \times 100 \times 6$$

$$=0.6/2[-0.0188 + 2(0.00785 + 0.0416$$

$$+0.00857 + 0.0416 + 0.00785) - 0.0188] \times 100 \times 6$$

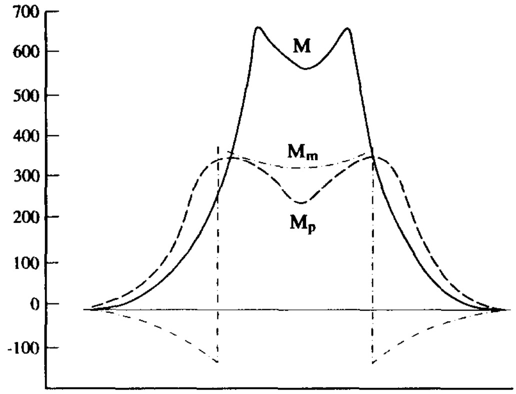
$$=59.69 \text{ kn m}.$$

3.4. *Solution to example* 4 (*Fig.* 5)

From the influence lines for $\beta = 0.2$ (Fig. 12), moment at C

$$=[(Im)_{3,1} \times M_1 + (Im)_{3,11} \times M_{11}]$$

† Area is computed using the trapezoidal rule with influence ordinates at every nodal point ($0.1L$).



M_p = Bending moment diagram due to external loads.

M_m = Bending moment diagram due to external moments.

M = Resultant bending moment diagram.

Fig. 7. Bending moment diagram.

$$=[(0.747 \times 1000) + (-0.0155)(-1000)]$$

$$=762.5 \text{ kN m}.$$

Moment at 0

$$=[(Im)_{6,1} \times M_1 + (Im)_{6,11} \times M_{11}]$$

$$=[(0.221 \times 1000) + (-0.221)(-1000)]$$

$$=442.0 \text{ kN m}.$$

3.5. *Solution to example* 5 (*Fig.* 6)

Non-dimensional parameter $\beta = 0.2$. Using the influence lines (Figs 9 and 12), the bending moments are computed at the nodal points and the bending moment diagram is drawn (Fig. 7).

From the influence coefficients, the bending moments are computed as

$$M_i = [(Im)_{i,m} \times P_m + (Im)_{i,n} \times P_n]$$

$$+[(Im)_{i,m} \times M_m + (In)_{i,n} \times M_n]$$

where $P_m = P_n = 1000$ kN and $M_m = 500$ kN m and $M_n = 500$ kN m.

3.6. *Comparison of results*

A comparison of the author's results and Hetenyi's classical solution is presented in Table 2.

From Table 2 it is evident that the author's results agree well with Hetenyi's classical solution.

Hetenyi's classical solution is not available for example 5 and therefore could not be compared.

4. CONCLUSIONS

(1) The computational procedure described in this paper to obtain the influence lines for bending

Table 1. Influence coefficients for example 5

Nodal points (i)	Influence coefficients (Fig. 9) for Pm $(Im)_{i,m}$	Influence coefficients (Fig. 9) for Pn $(Im)_{i,n}$	Influence coefficients (Fig. 12) for Mm $(Im)_{i,m}$	Influence coefficients (Fig. 12) for Mn $(Im)_{i,m}$
1	0.0055	−0.002	−0.05	−0.01
3	0.025	−0.004	−0.17	−0.06
4	0.0615	−0.00425	−0.35	−0.15
5	0.051	0.003	0.45	−0.23
6	0.02	0.02	0.29	−0.37
7	0.003	0.051	0.15	−0.53
8	−0.00425	0.0615	0.08	0.28
9	−0.004	0.025	0.03	0.14
10	−0.002	0.0055	0.0	0.04

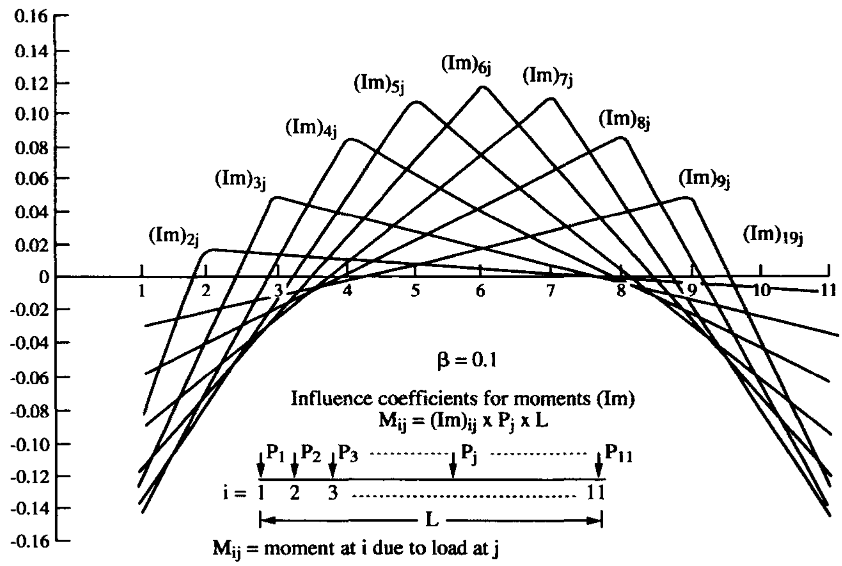


Fig. 8. Influence coefficients for moments due to external loads (for $\beta = 0.1$).

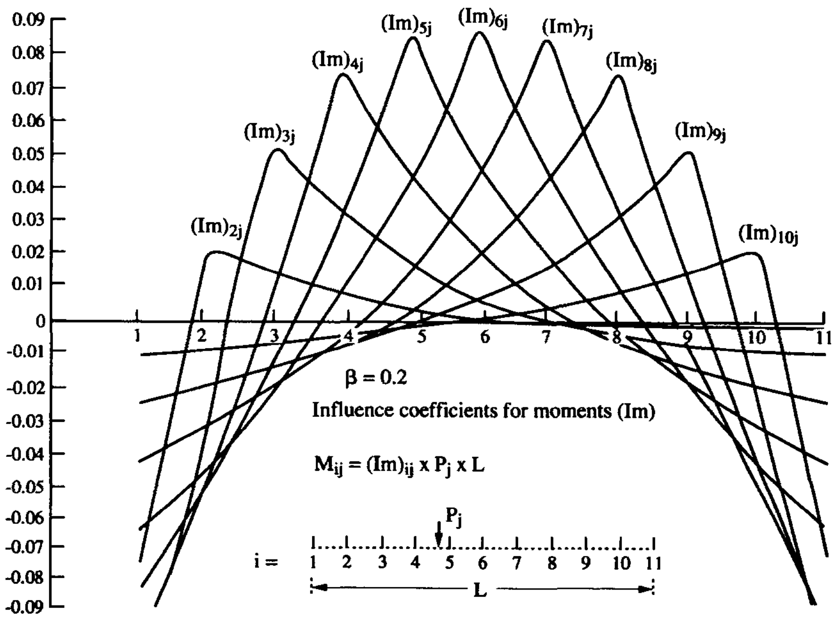


Fig. 9. Influence coefficients for moments due to external loads (for $\beta = 0.2$).

Table 2. Comparison of results

Example	1		2		3	4	
Moments at	C/D	0	C/D	0	0	C/D	0
Author	261.0	−225.6	260.8	−150.0	59.69	443.1	766.1
Hetenyi's (1)	262.5	−225.5	260.1	−148.1	60.30	442.0	762.5

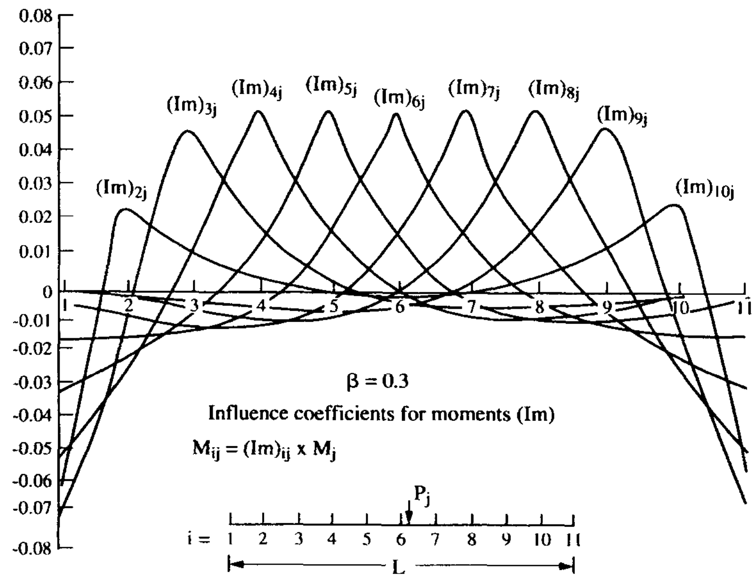


Fig. 10. Influence coefficients for moments due to external loads (for $\beta = 0.3$).

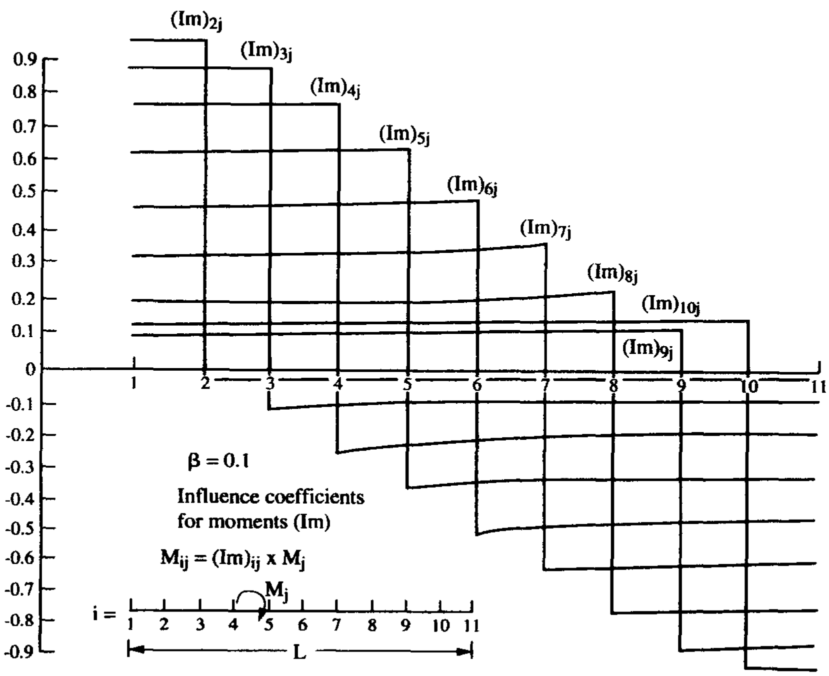


Fig. 11. Influence coefficients for moments due to external moments (for $\beta = 0.1$).

moments in beams on elastic foundations is very simple and is easily comprehensible.

(2) Influence lines are presented for the unit load and unit length of a beam for the non-dimensional parameters, $\beta = 0.1$, 0.2 and 0.3. These influence lines can easily be used to compute the bending moments in beams on elastic foundations, for any combination of load acting anywhere on the beam.

(3) Examples 1–5 demonstrate that a single influence line diagram can provide solutions for any type

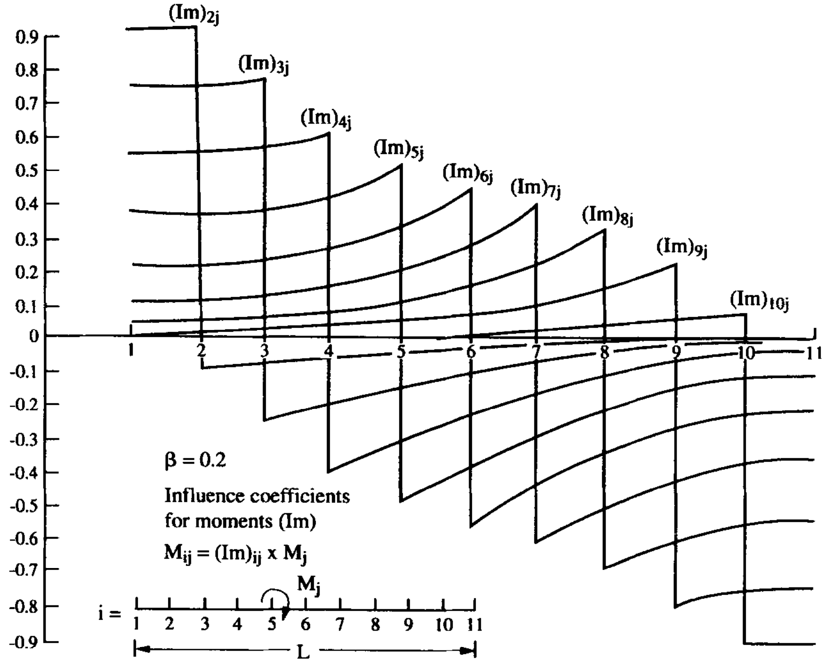


Fig. 12. Influence coefficients for moments due to external moments (for $\beta = 0.2$).

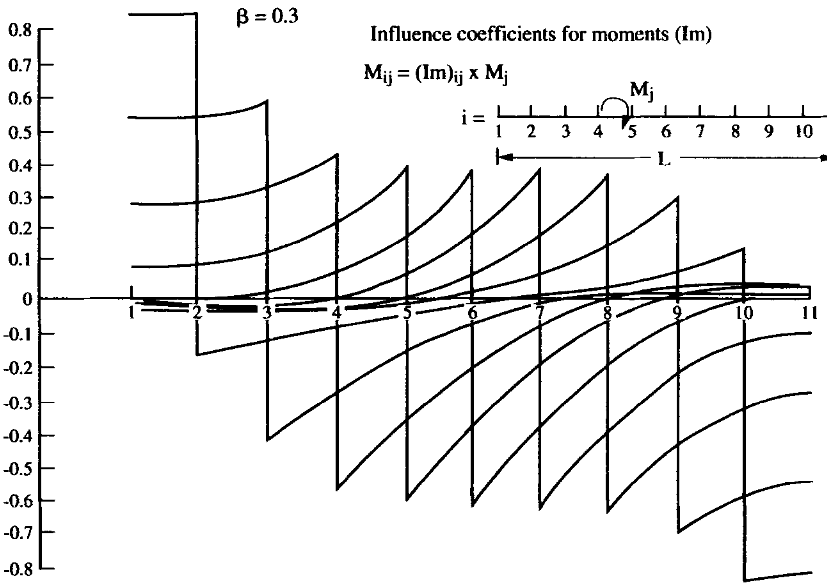


Fig. 13. Influence coefficients for moments due to external moments (for $\beta = 0.3$).

of load at any position, whereas, Hetenyi has provided separate solutions for each of these problems. The author's results are compared with Hetenyi's solutions (for those problems for which Hetenyi's solutions are available) and they agree well.

(4) The computational procedure presented can also be used for beams of varying cross-sections and soils of varying modulus of subgrade reaction, by imposing appropriate values of β in the set of finite difference equations.

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