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Flow of viscous stratified fluid of variable viscosity past a porous bed

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ABSTRACT

To study the effects of stratification and slip velocity on the flow of fluid of variable viscosity over a permeable bed, we divide the flow into two zones called zone 1 and zone 2. Zone 1 pertains to the flow called the free flow governed by the Navier-Stokes equations in the region between the impermeable upper plate and the porous bed. Zone 2 pertains to the flow in the bed governed by the modified Darcy Law. Using the slip velocity boundary condition, velocity distributions in zones 1 and 2 are obtained and are matched at the interface. The boundary layer thickness just beneath the permeable interface and the friction factor are also obtained.

1. INTRODUCTION

THE aim of this study is to investigate the flow of a viscous stratified fluid past a permeable bed with a motivation that stratification may provide a technique for studying pore size in a porous medium. The physical reason is that the stratification may retard or accelerate the flow depending on the magnitude of the stratification factor. The magnitude of retardation or acceleration is related to the slip parameter α, stratification factor γ and the porosity factor σ. Hence one would expect that these factors might provide a technique for studying pore size in a porous medium, which is very useful in petroleum industry in studying the factors which influence oil recovery from petroleum reservoirs. The earlier works by Beavers and Joseph,[1] Beavers *et al.*,[2] Rajasekhara *et al.*[4] and Rudraiah *et al.*[5] are all connected with the flow past a porous medium without stratification. It is well known that viscosity in oil varies with temperature and hence if the results are of some use in petroleum industry, one has to take the variation of viscosity of the fluid into consideration. Since the flow behaviour of fluids in petroleum reservoir rock depends to a large extent on the viscous stratification and porous properties of the rock, techniques of core study that

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yield new or additional information on the characteristics of the rock would contribute to a better understanding of petroleum reservoir performance. Therefore this paper concentrates on finding out the effect of viscous stratification on the flow past a porous bed.

When a Newtonian fluid flows between impermeable surfaces the usual boundary condition is the no-slip condition on the boundary which leads to a parabolic type of velocity profile in the channel. However, recent experiments[1,2,5] involving laminar flow of oil in flat rectangular ducts having one porous wall demonstrated the existence of a streamwise slip velocity at the nominal surface. Using this slip boundary condition we propose to investigate the laminar flow in a parallel plate channel having one permeable bounding wall. Denoting ' zone 1 ' for the free flow above the bed and ' zone 2 ' for the Darcy flow below the bed, the basic equations. and relevant boundary conditions in these zones are developed in section 2 In section 3, the velocity distributions in these two zones are separately obtained and matched at the interface to get a continuous velocity distribution. The mass flow rate and friction factors are also obtained in section 3. Expressions for the slip parameter α and the boundary layer thickness δ are obtained in section 4. Section 5 is devoted to a general discussion of the results.

2. MATHEMATICAL FORMULATION

The physical model shown in figure 1 consists of two zones. In zone one, from the impermeable upper rigid plate up to the interface, the flow called the free flow, is governed by the usual Navier-Stokes equations. In the other zone, below the interface, the flow is governed by the Darcy Law. In the following discussion we shall refer to these zones as ' zone 1 ' and ' zone 2 ' respectively.

The basic equations for zone 1 are

$$(\mu U')' = \partial p/\partial x, \tag{2.1}$$

the prime denoting differentiation with respect to y.

$$\mu = \mu_0 e^{-\beta y} \qquad \text{and} \qquad \rho = \rho_0 e^{-\beta y} \tag{2.2}$$

$$\partial p/\partial y = \rho g \tag{2.3}$$

where μ_0 and ρ_0 are the coefficients of viscosity and density respectively at the interface $y = 0$, and $\beta > 0$ represents the stratification factor.

The basic equations for ' zone 2 ' are

$$Q = Q_0 e^{\beta y} \tag{2.4}$$

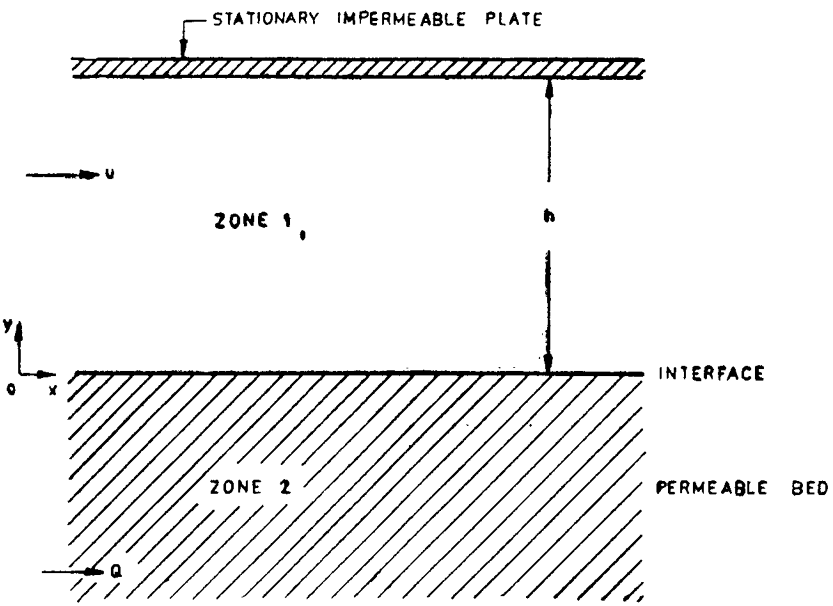

Figure 1. Physical model.

where

$$Q_0 = -K(\partial p/\partial x)/\mu_0. \tag{2.5}$$

These equations have to be solved using the following boundary conditions:

$$U = 0 \text{ at } y = h \tag{2.6}$$

$$U' = \alpha(U_B - Q_0)/\sqrt{K} \quad \text{at} \quad y = 0 \tag{2.7}$$

where α is the slip parameter, K is the permeability coefficient (has the dimension of length square), U_B is the slip velocity at the nominal surface $y = 0$ and Q_0 is given by (2.5). This slip-boundary condition, which is analogous to the slip-condition in the kinetic theory of gases, was first postulated by Beavers and Joseph[1] and for which a rigorous theoretical justification was given later by Saffman.[7] The effect of this velocity slip is to cause a skewing of the main flow velocity profile in the channel. We find that when $K \to 0$, the slip boundary condition (2.7) reduces to the no-slip boundary condition $U = 0$.

3. Velocity Distribution

In this section, we determine the velocity distribution in 'zone 1' using the boundary conditions (2.6) and (2.7). For this eq. (2.1) is made dimensionless using the quantities

$$(v, \pi, \eta, \xi) = (U/U_m, p/\rho_0 U_m^2, y/h, x/h) \tag{3.1}$$

and obtain

$$v'' - \gamma v' = - P e^{\gamma\eta} \tag{3.2}$$

and the corresponding boundary conditions are

$$v = 0 \quad \text{at} \quad \eta = 1 \tag{3.3}$$

$$v' = \alpha\sigma\,[v_B - (P/\sigma^2)] \text{ at } \eta = 0 \tag{3.4}$$

the prime denoting differentiation with respect to η, where

$$[\gamma, R, \sigma, D, P] = [\beta h, UD/v_0, h/\sqrt{k}, 2h, - R\,(\partial\pi/\partial\xi)/2].$$

Here γ is the non-dimensional stratification factor, R is the Reynolds number and U_m is the maximum velocity of the flow.

Solution of (3.2), which satisfies the conditions (3.3) and (3.4), is

$$v = \left[\frac{P}{\gamma^2} + \frac{\alpha\sigma}{\gamma} v_B - \frac{P}{\sigma^2}\right](e^{\gamma\eta} - e^{\gamma}) + \frac{P}{\gamma}(e^{\gamma} - \eta e^{\gamma\eta}) \tag{3.5}$$

where v_B is the dimensionless slip velocity at the nominal surface. Eq. (3.5) can be written in the form

$$v = v_B\left[\frac{e^{\gamma\eta} - e^{\gamma}}{1 - e^{\gamma}}\right] - \frac{P}{\gamma}\left[\frac{\eta e^{\gamma\eta} - e^{\gamma}}{(1 - e^{\gamma})} + \frac{(1 - \eta)\,e^{(1+\eta)\gamma}}{(1 - e^{\gamma})}\right] \tag{3.6}$$

v_B being given by, $v_B = AP/\gamma$, where

$$A = \frac{(\sigma - \alpha\gamma)(1 - e^{\gamma}) + \gamma\sigma e^{\gamma}}{\sigma\,[\gamma - \sigma\alpha\,(1 - e^{\gamma})\,]}.$$

This enables us (i) to visualise directly the effect of permeability of the bed on the velocity distribution, and (ii) to recognise that the slip velocity v_B is proportional to the pressure gradient.

The average dimensionless velocity is given by

$$\bar{v} = P\,(a_1 + a_2)/\gamma^3 \tag{3.7}$$

where

$$a_1 = 2\,(e^{\gamma} - 1) - 2\gamma e^{\gamma} + \gamma^2 e^{\gamma}$$

and

$$a_2 = \frac{\alpha}{\sigma}\left[\frac{\sigma^2 (1 - e^{\gamma}) + \gamma\sigma^2 e^{\gamma} - \gamma^2}{\gamma - \alpha\sigma(1 - e^{\gamma})}\right](e^{\gamma} - 1 - \gamma e^{\gamma}).$$

To find the quantitative effect of slip velocity and stratification factor, it is required to calculate the mass flow rate in the channel. If M denotes the dimensionless mass flow rate per unit channel width, then

$$M = \int_0^1 e^{-\gamma\eta}\, v d\eta = (P/\gamma^2)\,(1 - a\gamma/\sigma)\,(1 + 1/\gamma - e^{\gamma}/\gamma)$$
$$+ (P/\gamma)\,(e^{\gamma}/\gamma - 1/\gamma - \tfrac{1}{2}) + (v_B a\sigma/\gamma)\,(1 + 1/\gamma - e^{\gamma}/\gamma). \qquad (3.8)$$

Similarly, if M^* denotes the dimensionless mass flow rate when the porous bed is replaced by an impermeable rigid plate, then

$$M^* = -\frac{P}{\gamma^2}\left(1 + \frac{\gamma e^{\gamma}}{1 - e^{\gamma}} + \frac{\gamma}{2}\right). \qquad (3.9)$$

The mass flow rate through a porous walled channel and an impermeable walled channel may now be compared for the condition of equal pressure gradients and channel heights. From eqs (3.8) and (3.9), we get

$$\frac{M}{M^*} = 1 + \frac{2}{2 + 2\delta_0 e^{\gamma} + \gamma}\left[\frac{\gamma - \delta_0^2 e^{\gamma} - \delta_0}{\delta_0}\right.$$
$$\left. + \frac{(1 + \delta_0)\,(a\gamma - a\sigma^2 e^{\gamma} - \sigma)}{\sigma\delta_0 - a\sigma^2}\right] \qquad (3.10)$$

where

$$\delta_0 = \gamma/(1 - e^{\gamma}).$$

We note that in the limit $\gamma \to 0$, equation (3.10) reduces to

$$\mathop{\mathrm{Lt}}_{\gamma\to 0} \frac{M}{M^*} = 1 + \frac{3\,(\sigma + 2a)}{\sigma\,(1 + a\sigma)} \qquad (3.11)$$

which is the same as the result obtained by Beavers and Joseph.[1]

The fractional increase in mass flow rate through the channel with a permeable lower wall over what it would be if the wall were impermeable is

$$\Phi = (M - M^*)/M^*. \qquad (3.12)$$

The above theory is applicable only for laminar flow. Therefore, it is of interest to find the critical Reynolds number at which transition from laminar to turbulent flow occurs. To identify the break-down of the laminar flow regime for a fixed slip velocity ratio characterised by a fixed value of σ, we shall use the friction factor C_f defined by

$$C_f = -(\partial p/\partial x)\, D/[(1/2)\,\rho_0 \bar{U}^2] \qquad (3.13)$$

where D is the equivalent diameter and is equal to $2h$ for a parallel plate channel. Eq. (3.13), using eq. (3.7), becomes

$$C_f R = 8\gamma^3/(a_1 + a_2) \tag{3.14}$$

Where

$$R = \bar{U}D/\nu_0.$$

This $C_f R$ product is independent of Reynolds number for a channel of fixed height, fixed stratification factor and for a given boundary wall.

Similarly, the friction factor for a solid walled channel is

$$(C_f R)^* = 8\gamma/[(e^\gamma/(1 - e^\gamma)) - ((1 - e^\gamma)/\gamma^2)]. \tag{3.15}$$

Thus,

$$(C_f R)/(C_f R)^* = [(e^\gamma/(1 - e^\gamma)) - ((1 - e^\gamma)/\gamma^2)]\, \gamma^2/(a_1 + a_2). \tag{3.16}$$

4. EXPRESSIONS FOR BOUNDARY LAYER THICKNESS AND THE SLIP PARAMETER

Beavers and Joseph[1] have postulated the slip boundary condition (2.7) on the assumption that there exists a thin boundary layer just beneath the interface. It is of interest to find the expression for this boundary layer thickness, say δ. For this we have to use the boundary layer type of equation[3]

$$U'' - \beta U' - (1/K)\, U = e^{\beta y}\, (\partial p/\partial x)/\mu_0. \tag{4.1}$$

A solution of this equation satisfying the boundary conditions

$$U = U_B \quad \text{at} \quad y = 0 \tag{4.2}$$

$$U = Q_0\, e^{-\beta\delta} = -K\,(\partial p/\partial x)\, e^{-\beta\delta}/\mu_0 \text{ at } y = -\delta \tag{4.3}$$

is

$$U = (U_B - Q_0)\, e^{\beta y/2}\, (\cosh \lambda y + \coth \lambda\delta \cdot \sinh \lambda y) + Q_0\, e^{\beta y} \tag{4.4}$$

where

$$\lambda = (1/2)\sqrt{\beta^2 + (4/K)}$$

We know that at the edge of the boundary layer, the shear has to be zero. In other words

$$U' = 0 \quad \text{at } y = -\delta. \tag{4.5}$$

Then from eq. (4.4), using (4.5), we get

$$\lambda\,(U_B - Q_0) + \beta\, Q_0\, e^{-\beta\delta/2} \sinh \lambda\delta = 0. \tag{4.6}$$

This equation for δ is transcendental and it is difficult to obtain an analytical solution. However, we feel that since the boundary layer thickness and stratification factors are very small, we can neglect cubes and higher powers of δ and obtain

$$\frac{\delta}{h}=\frac{1}{\gamma}\left[1+\sqrt{1+\frac{2a_3}{\gamma\left[\gamma-a\sigma\left(1-e^{\gamma}\right)\right]}}\right] \tag{4.7}$$

which gives an expression for the boundary layer thickness, where

$$a_3=\sigma^2\left(1-e^{\gamma}\right)+\gamma\sigma^2 e^{\gamma}-\gamma^2.$$

The slip parameter α has to be determined experimentally. However, in certain cases[6,8,9] where the resistance of porous material would be regarded as a very sparse distribution of points at each of which the fluid exerts a force $(\mu/K)\ U$ where eq. (4.1) is true we can calculate the slip parameter α. For this we solve (4.1) using the boundary condition (4.2) and $U \rightarrow 0$ as $y \rightarrow -\infty$ and the resulting equation is

$$U=\left(U_B-Q_0\right) e^{(\beta/2+\lambda) y}+Q_0 e^{\beta y} \tag{4.8}$$

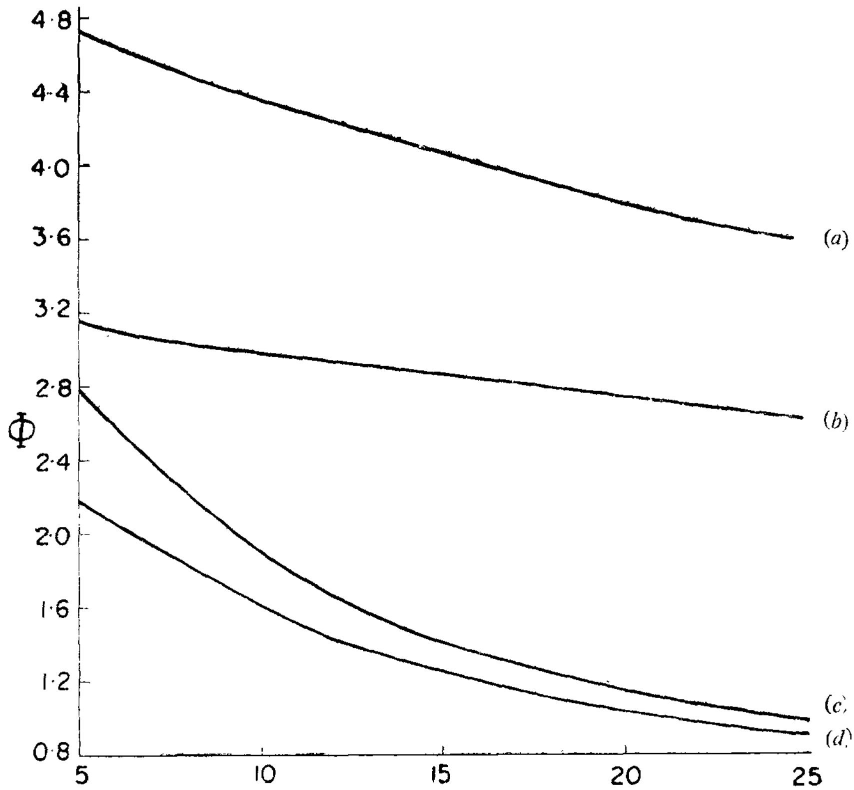

Figure 2. Fractional increase in mass flow rate (Φ) as a function of porosity factor (σ). (*a*) $\alpha=0{\cdot}01$, $\gamma=1$; (*b*) $\alpha=0{\cdot}01$, $\gamma=0{\cdot}2$; (*c*) $\alpha=0{\cdot}1$, $\gamma=1$; (*d*) $\alpha=0{\cdot}1$, $\gamma=0{\cdot}2$.

so that

$$(U')_{y=0} = [(\beta/2) + \lambda)(U_B) - Q_0] + Q_0\,\beta. \tag{4.9}$$

It may be noted that (4.9) is compatible with (2.7). Comparison of eq. (4.9) with eq. (2.7) for this model gives

$$\alpha = \frac{\sigma a_4 a_3 + \gamma^3}{\sigma a_3\left[1 + \dfrac{\sigma\gamma^3(1 - e^{\gamma})}{a_3}\right]} \tag{4.10}$$

where

$$a_4 = \frac{\gamma}{2\sigma} + \frac{1}{2\sigma}\sqrt{\gamma^2 + 4\sigma^2}\,.$$

We note that the expression for α tends to 1 as $\gamma \to 0$ which is the same as the one given by Taylor.[8]

5. DISCUSSION

To study the effect of slip velocity and viscous stratification factor on the flow over a permeable surface we divide the flow of fluid into two zones: zone 1 and zone 2. Zone 1 refers to the pressure flow, governed by the Navier-Stokes equations in the region between the impermeable upper plate and the porous bed. Zone 2 refers to the flow in the bed governed by the Darcy Law. Using the slip boundary condition the velocity distribution in zones 1 and 2 are obtained and are matched at the interface. To find the quantitative effect of slip velocity and stratification factor, the fractional increase, Φ, given by eq. (3.12) is numerically evaluated for different values of σ, α and γ and are shown in figures 2, 3 and 4.

From figure 2, it is clear that the fractional increase Φ decreases with increasing σ and increases with decreasing α. In other words the effect of slip at the bed is to increase the mass flow rate while the increase in permeability has the opposite effect.

Figures 3 and 4 pinpoint the variation of the fractional increase in mass flow rate with respect to the stratification factor γ. We find, as before, that Φ increases with decreasing σ and α where the growth remains linear up to about $\gamma = 1{\cdot}5$. This means that the viscosity stratification is favourable to the mass flow rate.

The slip parameter α as calculated from eq. (4.10) for different values of γ is shown in figure 5, which shows that the slip parameter increases with increasing γ and decreases with increasing σ.

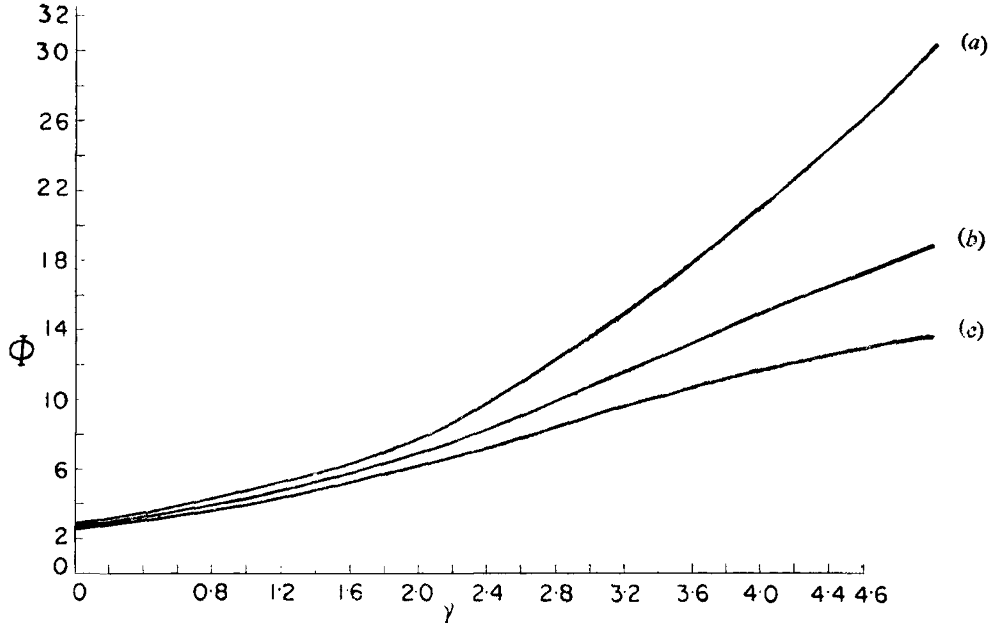

Figure 3. Fractional increase in mass flow rate (Φ) as a function of stratification fractor (γ). (a) $\alpha = 0 \cdot 01$, $\sigma = 5$; (b) $\alpha = 0 \cdot 01$, $\sigma = 10$; (c) $\alpha = 0 \cdot 01$, $\sigma = 15$.

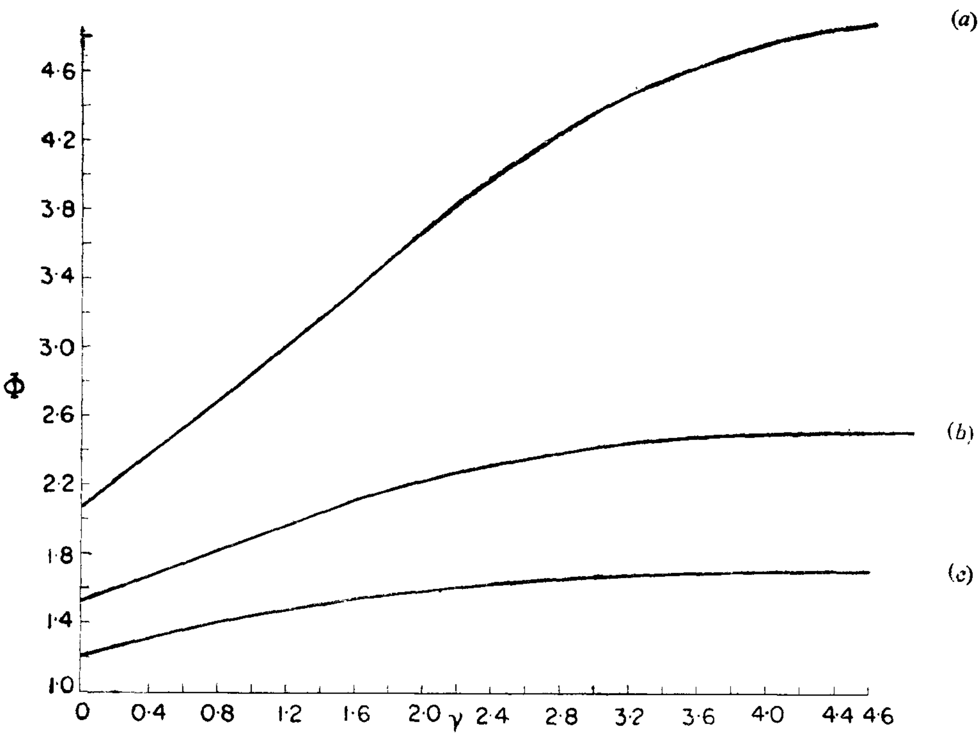

Figure 4. Fractional increase in mass flow rate (Φ) as a function of stratification factor (γ). (a) $\alpha = 0 \cdot 1$, $\sigma = 5$; (b) $\alpha = 0 \cdot 1$, $\sigma = 10$; (c) $\alpha = 0 \cdot 1$, $\sigma = 15$.

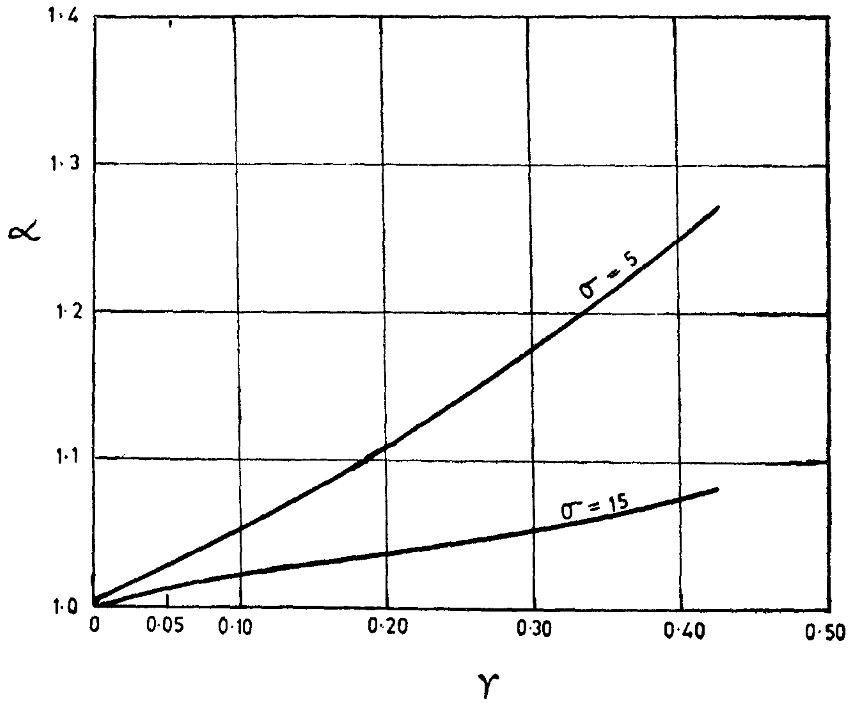

Figure 5. Slip parameters for different values of γ.

The verification of these conclusions by experiments will be reported in subsequent communications.

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