Extended And Unified Local Convergence For Newton-Kantorovich Method Under $w-$ Conditions With Applications

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Abstract: The goal of this paper is to present a local convergence analysis of Newton's method for approximating a locally unique solution of an equation in a Banach space setting. Using the gauge function theory and our new idea of restricted convergence regions we present an extended and unified convergence theory.

Key–Words: Newton's method, Banach space, semilocal convergence, gauge function, convergence region, Newton-Kantorovich theorem.

1 Introduction

Many problems in computational sciences and related areas can be formulated as equations defined on some spaces using Mathematical modeling [2,4,5,7,12,22]. The closed form solution of such equations is desirable but rarely attainable. That is why, in practice we utilize iterative methods. There is a plethora of local as well as semi-local convergence results for Newton's method under generalized Lipschitz-type conditions [1–49]. Newton's method defined for each $n = 0, 1, 2, \dots$ by

$$x_{n+1} = N(x_n), \tag{1.1}$$

is undoubtedly the most popular method for constructing a sequence $\{x_n\}$ approximating a solution p of nonlinear equation

$$F(x) = 0. \tag{1.2}$$

Here x_0 is an initial point, $F : D \subseteq \mathcal{B}_1 \longrightarrow \mathcal{B}_2, D$ is a convex subset of $\mathcal{B}_1; \mathcal{B}_1, \mathcal{B}_2$ are Banach spaces, F is a Fréchet-differentiable operator and $N(x) = x - F'(x)^{-1}F(x)$.

Proinov in [31] presented a generalization for the famous Banach contraction mapping principle with arbitrary order of convergence of Picard's iteration. Then, later in [33] some general local convergence results were reported for Picard's iteration with arbitrary order of convergence using an iteration function in a metric space. More recently, the semi-local convergence was given in [32] with order of convergence $\gamma \geq 1$ for Picard's iteration defined for each $n = 0, 1, 2, \dots$ by

$$x_{n+1} = P(x_n), \tag{1.3}$$

where $P : D \subset \mathcal{B}_1 \longrightarrow \mathcal{B}_1$ is an iteration function in $\mathcal{B}_1$ satisfying the condition

$$H(P(x)) \leq g(H(x)) \text{ for each } x \in D \tag{1.4}$$

with $P(x) \in D$ and $H(x) \in I$, $H : D \longrightarrow \mathbb{R}_+, I \subseteq \mathbb{R}_+$ containing 0 and g is a gauge function of I (to be precised in Section 2). $H : D \longrightarrow \mathbb{R}_+$ satisfying (1.4) is called a function of initial approximation of P. Using g, H and P the local convergence of Picard's method was given in [31]. These results were then specialized using $w-$versions of the Newton-Kantorovich theorem [22]. These results extended and unified earlier ones.

In the present study, we are motivated by work in [33] and optimization considerations. Using our new idea of restricted convergence regions we find a more precise location, where the Newton iterates lie than before. This way the resulting $w-$ functions are smaller than before leading to the following advantages ($\mathcal{A}$):

($\mathcal{A}_1$) Extended region of convergence leading to a wider choice of initial guesses.

($\mathcal{A}_2$) At least as precise error bounds on the distances $\|x_n - p\|$ leading to fewer iterates to obtain a desired error tolerance.

($\mathcal{A}_3$) An at least as precise information on the location of the solution p.

Advantages ($\mathcal{A}$) are obtained under the same computational effort, since in practice the computation of the $w-$function in [33] requires the computation of our new $w-$functions as special cases.

The rest of the study is structured as follows: Some Mathematical background on gauge functions and related materials is presented in Section 2. The local convergence of Newton's method is given in Section 3.

2 Gauge functions and initial conditions

In order to make the study as self contained as possible, we reproduce some gauge function related concepts. More details can be found in [32, 33].

Definition 2.1 *A function $g : I \longrightarrow \mathbb{R}_+$ is called quasi-homogeneous of degree $\delta \geq 0$ on I if it satisfies the condition*

$$g(\lambda t) \leq \lambda^\delta g(t) \text{ for each } \lambda \in (0,1) \text{ and } t \in I. \tag{2.1}$$

Definition 2.2 *A function $g : I \longrightarrow \mathbb{R}_+$ is called gauge function of order $\delta \geq 1$ on I if it is quasi-homogeneous of order δ on I and*

$$g(t) \leq t \text{ for each } t \in I. \tag{2.2}$$

Definition 2.3 *Let $P : D \subset \mathcal{B}_1 \longrightarrow \mathcal{B}_2$. A function $H : D \longrightarrow \mathbb{R}_+$ is called function of initial conditions of P with a gauge function g on I if there exists a function $g : I \longrightarrow I$ such that*

$$\begin{aligned} H(P(x)) \quad &\leq \quad g(H(x)) \text{ for each } x \in D \text{ with} \\ &P(x) \subset D \text{ and } H(x) \in I \end{aligned} \tag{2.3}$$

Definition 2.4 *Let $P : D \subset \mathcal{B}_1 \longrightarrow \mathcal{B}_1$ and let $H : D \longrightarrow \mathbb{R}_+$ be a function of initial conditions of P with a gauge function on I. Then a point $x \in D$ is called initial point of P if $H(x) \in I$ and iterates $P^n x$ are well-defined and remain in D.*

We shall need the following auxiliary result (see corollary 3.7 in [33]).

COROLLARY 2.5 *Let $T : D \subset X \longrightarrow X$ be an operator on a metric space (X, d) and let $\mu \in D$. Suppose that*

$$d(T(x), \mu) \leq \varphi(d(x, \mu)) \text{ for each}$$

$$x \in D \text{ with } d(x, \mu) \in I,$$

where φ is a strict gauge function of order $\delta \geq 1$ on I. Then, μ is a unique fixed point of T in the set $U = \{x \in D : d(x, \mu) \in I\}$. Moreover, if $T : U \longrightarrow U$, then for each $x_0 \in U$ the following items hold.

(i) Picard iteration $x_{n+1} = T(x_n)$ stays in U and converges to μ with $Q-$order δ.

(ii) $d(x_n, \mu) \leq \lambda^{s_n(\delta)} d(x_0, \mu)$ for each $n = 0, 1, 2, \ldots$ where $\lambda = \phi(E(x_0))$ and ϕ is a nondecreasing nonnegative function on I satisfying $\varphi(t) = t\phi(t)$ for each $t \in I$.

(iii) $d(x_{n+1}, \mu) \leq \varphi(d(x_n, \mu))$ for each $n = 0, 1, 2, \ldots$.

3 Local convergence analysis

The local convergence analysis of Newton's method is based on generalized $w-$ affine invariant conditions. Let p be such that $F(p) = 0$ and $F'(p)^{-1}$ exists. Suppose there exists function $w_0 : [0, +\infty) \longrightarrow [0, +\infty)$ continuous and increasing with $w_0(0) = 0$ such that for each $x \in D$

$$\|F'(p)^{-1}(F'(x) - F'(p))\| \leq w_0(\|x - p\|). \tag{3.1}$$

Moreover, suppose equation

$$w_0(t) = 1 \tag{3.2}$$

has positive solutions. Denote by ρ the smallest positive solution of equation (3.2). Set $D_0 = D \cap U(x_0, \rho)$. Furthermore, suppose there exists function $w : [0, 2\rho) \times [0, \rho] \times [0, \rho) \longrightarrow [0, +\infty)$ continuous and nondecreasing with $w(0, 0, 0) = 0$ such that for each $x, y \in D_0$

$$\|F'(p)^{-1}(F'(x) - F'(y))\| \leq w(\|x - y\|, \|x - p\|, \|y - p\|). \tag{3.3}$$

Notice that function w_0 depends on D, whereas w depends on w_0 and D_0. The construction of function w was not possible before, since w_0 and D_0 are needed [3, 9, 10, 13, 22, 33]:

$$\begin{aligned} &\|F'(p)^{-1}(F'(x) - F'(y))\| \\ \leq \quad &\bar{w}(\|x - y\|, \|x - p\|, \|y - p\|) \\ &\text{, for each } x, y \in D \end{aligned} \tag{3.4}$$

where $\bar{w} : [0, +\infty)^3 \longrightarrow [0, +\infty)$ is a continuous and nondecreasing function with $\bar{w}(0, 0, 0) = 0$. We have that for $s, s_1, s_2, s_3 \geq 0$

$$w_0(s) \leq \bar{w}(s, s, s) \tag{3.5}$$

and

$$w(s_1, s_2, s_3) \leq \bar{w}(s_1, s_2, s_3) \tag{3.6}$$

since $D_0 \subseteq D$ and $\frac{\bar{w}}{w_0}$ can be arbitrarily large [3, 4]. Let us provide a motivational example to show that (3.5) and (3.6) can hold as strict inequalities. Let $\mathcal{B}_1 =$

$\mathcal{B}_2 = \mathbb{R}^3, D = \bar{U}(0,1)$ and $p = (0,0,0)^T$. Define function F on D for $v = (v_1, v_2, v_3)^T$ by

$$F(v) = (e^{v_1} - 1, \frac{e-1}{2}v_2^2 + v_2, v_3)^T. \quad (3.7)$$

The Fréchet derivative is given by

$$F'(v) = \begin{bmatrix} e^{v_1} & 0 & 0 \\ 0 & (e-1)v_2 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, conditions (3.1), (3.3) and (3.4) are satisfied for $w_0(t) = L_0 t, w(t,t,t) = e^{\frac{1}{L_0}} t, \bar{w}(t,t,t) = Lt, \rho = \frac{1}{L_0}, L_0 = e - 1$ and $L = e$. Notice that

$$w_0(t) < w(t,t,t) < \bar{w}(t,t,t) \quad (3.8)$$

holds for all $t \in [0, \rho)$. The upper bound on the expression appearing in these studies using (3.4) is

$$\begin{aligned} & \|F'(p)^{-1}(F(x) - F(p) - F'(x)(x-p))\| \\ \leq & \|\int_0^1 F'(p)^{-1}(F'(x) - F'(p + \theta(x-p))) \\ & \times (x-p)d\theta\| \\ \leq & \int_0^1 \bar{w}(\|x-p\|, \theta\|x-p\|, (1-\theta)\|x-p\|)d\theta \\ = & \int_0^{\|x-p\|} \bar{w}(\|x-p\|, u, \|x-p\| - u)du \quad (3.9) \end{aligned}$$

However, using the more precise and actually needed condition (3.3) instead of (3.4) we obtain instead of (3.9) the more precise estimate (see also (3.6))

$$\begin{aligned} & \|F'(p)^{-1}(F(x) - F(p) - F'(x)(x-p))\| \\ \leq & \|\int_0^1 F'(p)^{-1}(F'(x) - F'(p + \theta(x-p))) \\ & \times (x-p)d\theta\| \\ \leq & \int_0^1 w(\|x-p\|, \theta\|x-p\|, (1-\theta)\|x-p\|)d\theta \\ = & \int_0^{\|x-p\|} w(\|x-p\|, u, \|x-p\| - u)du. (3.10) \end{aligned}$$

Replacing (3.4) and (3.9) by (3.3) and (3.10), respectively in the proof of the local convergence analysis of Newton's method in [33] we obtain a finer convergence analysis with advantages as already stated in the introduction of this study. The advantages are obtained under the same computational effort as in [33], since in practice the computation of function $\bar{w}$ requires the computation of function w as a special case.

The results in this Section, in particular reduce to the corresponding ones in Section 7 of [33], if $w_0 = w = \bar{w}$. Otherwise, the new results constitute an improvement (see also advantages $(\mathcal{A})$).

LEMMA 3.1 *Let $F : D \subset \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a continuously Fréchect differentiable operator on an open convex set D. Let $p \in D$ be a simple zero of F and the condition (3.1) and (3.3) are satisfied. Assume that the real function*

$$\varphi(t) = \frac{\int_0^1 w(t, u, t-u)du}{1 - w_0(t)} \quad (3.11)$$

is well-defined on the interval $(0, \rho)$. Then

$$\|N(x) - p\| \leq \varphi(\|x-p\|) \text{ for each } x \in D \text{ with } \|x-p\| < \rho. \quad (3.12)$$

Proof. Let $x \in D$ be such that $\|x - p\| < \rho$. Notice that since function φ is well-defined on $[0, \rho)$, then $w_0(t) < 1$ for each $t \in [0, \rho)$. Using this and (3.1), we get in turn that

$$\|F'(p)^{-1}(F'(x) - F'(p))\| \leq w_0(\|x-p\|) < 1,$$

since $0 \leq \|x - p\| < \rho$. It then follows from the preceding inequality and the Banach Lemma on invertible operators [5, 22] that $F'(x)^{-1}$ exists and

$$\|F'(x)^{-1}F'(p)\| \leq \frac{1}{1 - w_0(\|x-p\|)}.$$

Moreover, we have that

$$\begin{aligned} & \|T(x) - p\| \\ \leq & \|F'(x)^{-1}F'(p)\| \\ & \times \|\int_0^1 F'(p)^{-1}(F'(x) - F'(p + \theta(x-p))) \\ & (x-p)d\theta\| \\ \leq & \|F'(x)^{-1}F'(p)\| \\ & \times \int_0^1 w(\|x-p\|, \|\theta(x-p)\|, \\ & \|x - (p + \theta(x-p))\|)d\theta \\ \leq & \|F'(x)^{-1}F'(p)\| \\ & \times \int_0^{\|x-p\|} w(\|x-p\|, u, \|x-p\| - u)d\theta. \end{aligned}$$

The proof is completed if we combine the last two inequalities.

□

THEOREM 3.2 *Let $F : D \subset \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a continuously Fréchect differentiable operator on an open ball $U(p, \rho) \subset D$, where p is a simple zero of F. Suppose F' satisfies conditions (3.1) and (3.3) and the real function φ defined by (3.11) is strict gauge function of order $\xi + 1$ for some $\xi \geq 0$ on the interval $[0, \rho)$. Then for each $x_0 \in U(p, \rho)$ the following items hold true:*

(i) Newton iteration (1.1) is well-defined, remains in $U(p,\rho)$ and converges to p with $Q-$order $\xi+1$.

(ii) For all $n \geq 0$ we have the following estimate

$$\|x_{n+1}-p\| \leq \varphi(\|x_n-p\|). \tag{3.13}$$

(iii) For all $n \geq 0$ we have the following estimate

$$\|x_n-p\| \leq \lambda^{s_n(p+1)}\|x_0-p\|, \tag{3.14}$$

where $\lambda=\phi(\|x_0-p\|)$ and ϕ is a nondecreasing function on I satisfying $\varphi(t)=t\phi(t)$.

(iv) If ρ is a fixed point of φ, then ρ is the optimal radius of the convergence ball of Newton's method under the condition (3.1) and (3.3) for some w and w_0.

Proof. Items (i)-(iii) follow from Corollary 2.5 and Lemma 3.1. Item (iv) is shown in Theorem 4.1.

□

4 Special cases

We assume from now on that

$$w_0(t) \leq w_1(t) \;\text{ for }\; t\in[0,\rho). \tag{4.1}$$

If

$$w_1(t) \leq w_0(t) \;\text{ for }\; t\in[0,\rho), \tag{4.2}$$

where w_1 is a nonnegative nondecreasing function of $[0,\rho)$. Then, the results that follow hold with w_0 replacing w_1. In the following theorems and corollaries in this section we consider some interesting special cases of Theorem 3.2.

THEOREM 4.1 *Let $F: D\subset \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a continuously Fréchect differentiable operator on an open ball $U(p,\rho)\subset D$, where p is a simple zero of F. Suppose*

$$\|F'(p)^{-1}(F'(x)-F'(y))\| \leq w_1(\|x-p\|)-w_1(\|y-p\|) \tag{4.3}$$

for all $x,y\in U(p,\rho)$, where w_1 is a real function defined on $[0,\rho]$ with $w_1(0)=0$. Moreover, suppose that (3.1) holds. Furthermore, suppose that

$$\varphi(t)=\frac{tw_1(t)-\int_0^1 w_1(u)du}{1-w_0(t)} \tag{4.4}$$

is a strict gauge function of order $\xi+1$ for some $p\geq 0$ on $[0,r)$. Then starting from every $x_0\in U(p,\rho)$ Newton iteration (1.1) is well defined, remains in $U(p,\rho)$, converges with $Q-$order $\xi+1$ to p and satisfies the estimates (4.1) and (4.4). Moreover, if ρ is a fixed point of φ and w_0,w_1 are continuous, then ρ is the optimal radius of the convergence ball of Newton's method.

Proof. The first part of the Theorem follows immediately from Theorem 3.2. Let ρ be a fixed point of φ. We shall show the exactness of ρ even if $\mathcal{B}_1=\mathcal{B}_2=\mathbb{R}$. Let $p\in\mathbb{R}_+$ be a arbitrary. Define function F on $D=\bar{U}(p,\rho)$ by

$$F(x)=x-p-sign(x-p)\int_0^{\|x-p\|} w_1(t)dt,$$

where w_1 is continuous on $[0,\rho)$ with $w_1(0)=0$. Clearly, F is continuously differentiable with $F'(x)=1-w_1(|x-p|)$. Then, for all $x\in U(p,\rho)$ and $y\in[p,x]$, we get

$$|F'(p)^{-1}(F'(x)-F'(y))\| = w_1(|x-p|)-w_1(|y-p|)$$

which shows that (4.3) holds. Then, $T(p+\rho)=p-\varphi(\rho)=p-\rho$ and $T(p-\rho)=p+\varphi(\rho)=p+\rho$, for $T(x)=x-F'(x)^{-1}F(x)$. Hence, if $x_0=p+\rho$, then $x_n=p+(-)^n\rho$. Therefore, Newton's method starting at $x_0=p+\rho$ fails to converge.

□

REMARK 4.2 *Let us give a sufficient condition for φ defined by (4.4) to be a gauge function of order $\xi+1$. It follows from the Example in section 3 that if w_1 is a nonnegative nondecreasing function on $[0,\rho)$ such that for all $\lambda\in(0,1)$ and all $t,u\in[0,\rho)$ with $t\geq u$ it satisfies $w_1(\lambda t)-w_1(\lambda u)\leq \lambda^{\xi}[w_1(t)-w_1(u)]$ for some $\xi\geq 0$, then the function φ defined by (4.4) is a strict gauge function of order $\xi+1$ on I provided that $\varphi(t)<t$ for all $t\in(0,\rho)$.*

Note that in the case $w_0(t0=w_1(t)=Lt$ condition (4.3) coincides with (4.2) and we obtain Traub and Woźniakowski's result [39]. By putting $w_1(t)=\int_0^t L(u)du$, where L is nondecreasing on $[0,\rho)$, we immediately get some results of Wang [43, Theorem 3.1 and 5.1]. Theorem 4.1 is also an improvement of a result by Wang and Li [45, Theorem 1.1] and Proinov [33].

COROLLARY 4.3 *Let $F: D\subset\mathcal{B}_1\longrightarrow\mathcal{B}_2$ be a continuously Fréchect differentiable operator on an open ball $U(p,\rho)\subset D$, where p is a simple zero of F. Suppose (4.3) holds with*

$$\begin{aligned} w_0(t) &= L_0t^{\xi}, \\ w_1(t) &= L_1t^{\xi} \; (\xi>0.L_0,L_1>0) \text{ and} \\ 0<r &\leq R=\left(\frac{\xi+1}{\xi L_1+(\xi+1)L_0}\right)^{1/\xi}. \end{aligned} \tag{4.5}$$

Define the real functions

$$\varphi(t)=\frac{\xi}{\xi+1}\frac{L_1t^{\xi+1}}{1-L_0t^{\xi}} \text{ and } \phi(t)=\frac{\xi}{\xi+1}\frac{L_1t^{\xi}}{1-L_0t^{\xi}}. \tag{4.6}$$

Then starting from every $x_0 \in U(p,\rho)$ Newton iteration (1.1) is well defined, remains in $U(p,\rho)$, converges with $Q-$order $\xi+1$ to p and satisfies the estimates (4.1) and (4.4). Moreover, if $\rho = R$ is the optimal radius of the convergence ball of Newton's method under the condition (4.3) with w_1 defined by (4.5).

In the case when $\xi = 1$ and $L_0 = L_1$, we again get the above mentioned result of Traub and Woźniakowski [39] as well as the result of Rheinboldt [35], Wang [42] and Ypma [49]. If $0 < \xi \leq 1$, then Corollary 4.3 leads to the results of Wang and Li [45, Corollary 3.1], Huang [19, Theorem 2] and Proinov [33]. If $L_0 < L_1$ the mentioned results are improved.

COROLLARY 4.4 *Let $F : D \subset \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a continuously Fréchect differentiable operator on an open ball $U(p,\rho) \subset D$, where p is a simple zero of F. Suppose that (3.1) and (4.3) hold with*

$$\begin{aligned} w_0(t) &= \frac{c_0}{(1-\gamma_0 t)^2}, \qquad (4.7)\\ w_1(t) &= \frac{c}{(1-\gamma t)^2}, (\gamma > 0, L_0, L_1 > 0) \text{ with}\\ c_0 &\leq c \text{ and } \gamma_0 \leq \gamma \end{aligned}$$

Define the real function

$$\Phi(t) = \frac{\varphi(t)}{t} \qquad (4.8)$$

and

$$0 < r \leq R, \qquad (4.9)$$

where R is the smallest positive solution of $\varphi(t) = t$. Then starting from every $x_0 \in U(p,\rho)$ Newton iteration (1.1) is well defined, remains in $U(p,\rho)$, converges with $Q-$quadratically to p with the estimates (4.1) and

$$\|x_n - p\| \leq \lambda^{2^n-1}\|x_0 - p\| \text{ for all } n \geq 0, \qquad (4.10)$$

where $\lambda = \Phi(\|x_0 - p\|)$. Moreover, if $\rho = R$ is the optimal radius of the convergence ball of Newton's method under the conditions (3.1) and (4.3).

Corollary 4.4 without the estimate (4.1) and for $L_0 = L, \gamma_0 = \gamma$ is due to Wang and Han [44] ($L = 1$)and Wang [43, Example 1]($L \geq 0$). Note that the function Φ defined in 4.8) is strictly increasing and continuous on the interval $I = [0,R]$ and satisfies $\Phi(I) = [0,1]$. Therefore, if we ignore the estimate (4.1), then Corollary 4.4 is equivalent to the following result of Wang [43, p.132]. However, if $L_0 < L$ and $\gamma_0 < \gamma$ the mentioned results are improved.

COROLLARY 4.5 *Let $F : D \subset \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a continuously Fréchect differentiable operator on an open ball $U(p,\rho) \subset D$, where p is a simple zero of F. Suppose that (3.1) and (4.3) hold and R satisfies (4.9). Then starting from every $x_0 \in U(p,\rho)$ Newton iteration (1.1) is well defined, remains in $U(p,\rho)$, converges with $Q-$quadratically to p with the estimates (4.1) and*

$$\|x_n - p\| \leq \lambda^{2^n-1}\|x_0 - p\| \text{ for all } n \geq 0. \qquad (4.11)$$

The following theorem is another natural generalization of Traub and Woźniakowski's result [39] mentioned in the beginning of the section.

THEOREM 4.6 *Let $F : D \subset \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a continuously Fréchect differentiable operator on an open ball $U(p,\rho) \subset D$, where p is a simple zero of F. Suppose that (3.1) and*

$$\|F'(p)^{-1}(F'(x) - F'(y))\| \leq w_2(\|x-y\|) \qquad (4.12)$$

for all $x, y \in U(p,\rho)$ where w_2 is a real function defined on $[0,\rho]$ with $w_2(0) = 0$. Assume that

$$\varphi(t) = \frac{\int_0^t w_2(u)du}{1 - w_0(t)} \qquad (4.13)$$

is a strict gauge function of order $\xi+1$ for some $0 \leq \xi \leq 1$ on $[0,\rho)$. Then starting from every $x_0 \in U(p,\rho)$ Newton iteration (1.1) is well defined, remains in $U(p,\rho)$, converges with $Q-$order $\xi+1$ to p and with the error estimates (4.1) and (4.5).

COROLLARY 4.7 *Let $F : D \subset \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a continuously Fréchect differentiable operator on an open ball $U(p,\rho) \subset D$, where p is a simple zero of F. Suppose conditions (3.1) and (4.12) hold with*

$$\begin{aligned} w_0(t) &= L_0 t^\xi, \qquad (4.14)\\ w_2(t) &= L_2 t^\xi,\ (0 \leq \xi \leq 1, L_0, L_2 > 0) \text{ and}\\ 0 < r &\leq R = \left(\frac{\xi+1}{\xi L_2 + (\xi+1)L_0}\right)^{1/\xi}. \end{aligned}$$

In the case $\xi = 0$ we assume that $L_0 < \frac{1}{2}$ and $R = \infty$. Define functions

$$\varphi(t) = \frac{\xi}{\xi+1}\frac{L_2 t^{\xi+1}}{1 - L_0 t^\xi} \text{ and } \phi(t) = \frac{\xi}{\xi+1}\frac{L_2 t^\xi}{1 - L_0 t^\xi}. \qquad (4.15)$$

Then starting from every $x_0 \in U(p,\rho)$ Newton iteration (1.1) is well defined, remains in $U(p,\rho)$, converges with $Q-$order $\xi+1$ to ξ and satisfies the estimates (4.1) and (4.4).

From Corollary 4.7 in the case $\xi = 1$ we again get the classical results of Rheinboldt [35, 36], Traub and Woźniakowski [39], Wang [42–45], Ypma [49]. In the case $0 \leq \xi \leq 1$ Corollary 4.7 is obtained by Ypma [49, Theorem 3.1], Huang [19, Theorem 1] and in a slightly different form by Argyros [6, Theorem 4].

THEOREM 4.8 *Let $F : D \subset \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a continuously Fréchect differentiable operator on an open ball $U(p, \rho) \subset D$, where p is a simple zero of F. Suppose that (3.1) and*

$$\|F'(p)^{-1}(F'(x) - F'(y))\| \leq w_3(\|x - y\|) \quad (4.16)$$

for all $x, y \in U(p, \rho)$, where w_3 is a real function defined on $[0, \rho]$ with $w_3(0) = 0$. Assume that

$$\varphi(t) = \frac{tw_3(t) + \int_0^t w_3(u)du}{1 - w_0(t)} \quad (4.17)$$

is a strict gauge function of order $\xi + 1$ for some $0 \leq \xi \leq 1$ on $[0, \rho)$. Then starting from every $x_0 \in U(p, \rho)$ Newton iteration (1.1) is well defined, remains in $U(p, \rho)$, converges with $Q-$order $\xi + 1$ to p and with the error estimates (4.1) and (4.5).

REMARK 4.9 *Let φ be defined by (4.15) or (4.18). It follows from the Example 2.2 in [33] that if w_3 is nonnegative nondecreasing on $[0, \rho)$ and $\frac{w_3(t)}{t^p}$ is nondecreasing on $(0, \rho)$ for some $\xi \geq 0$, then φ is a strict guage function of order $\xi + 1$ on $[0, \rho)$ provided that $\varphi(t) < t$ for all $t \in (0, \rho)$.*

COROLLARY 4.10 *Let $F : D \subset \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a continuously Fréchect differentiable operator on an open ball $U(p, \rho) \subset D$, where p is a simple zero of F. Suppose that (4.16) hold with*

$$\begin{aligned} w_0(t) &= L_0 t^{\xi}, \quad (4.18)\\ w_3(t) &= L_3 t^{\xi}, (\xi > 0.L_0 > 0) \text{ and}\\ 0 &< \rho \leq R = \left(\frac{\xi + 1}{\xi L_3 + (\xi + 1)L_0}\right)^{1/\xi} \end{aligned}$$

In case $\xi = 0$ we assume That $L_0 < \frac{1}{3}$ and $R = \infty$. Define the real functions

$$\varphi(t) = \frac{(\xi + 2)L_3 t^{\xi+1}}{(\xi + 1)(1 - L_0 t^{\xi})} \text{ and } \phi(t) = \frac{(\xi + 2)L_3 t^{\xi}}{(\xi + 1)(1 - L_0 t^{\xi})}. \quad (4.19)$$

Then starting from every $x_0 \in U(p, \rho)$ Newton iteration (1.1) is well defined, remains in $U(p, \rho)$, converges with $Q-$quadratically to p with the estimates (4.3)and (4.5).

From Corollary 4.10 in the case $\xi = 1$ we obtain $R = \frac{2}{5L}$ which improves a recent result of Wang and Li [45, Corollary 3.2]. For example they have proved $R \leq \frac{1}{3L_0}$, since Proinov [33] $R_{\xi} = \frac{2}{5L} < \frac{2}{5L_0}$, The following corollary is an improvement of another result of Wang and Li [45, Corallary 3.3].

COROLLARY 4.11 *Let $F : D \subset \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a continuously Fréchect differentiable operator on an open ball $U(p, \rho) \subset D$, where p is a simple zero of F. Suppose that (4.16) holds with w_3 defined by (4.18) and R satisfies (4.18). Then starting from every $x_0 \in U(p, \rho)$ Newton iteration (1.1) is well defined, remains in $U(p, \rho)$, converges with $Q-$quadratically to p with the estimates (4.3)and (4.5).*

In the convergence ball of Newton's method, the solution p of the equation $F(x) = 0$ is certainly unique. But it is well known that the uniqueness ball of this equation may be larger. Such results can be found in [6, 33] using only (3.1).

EXAMPLE 4.12 *Returning back at the motivational example (3.7), we see by Lemma 3.1 and Theorem 3.2 that our new radius of convergence is $\rho_3 = \frac{2}{2(e-1)+e^{\frac{1}{e-1}}} = 0.3826919122323857$ the radius in [3, 4, 6] is $\rho_2 = \frac{2}{2(e-1)+e} = 0.324947231372689$ and the one in [19, 33, 39, 41] is $\rho_1 = \frac{2}{3e} = 0.2452529607809$, so $\rho_1 < \rho_2 < \rho_3$ justifying advantages $(\mathcal{A})$.*

EXAMPLE 4.13 *Let $\mathcal{B}_1 = \mathcal{B}_2 = C[0, 1]$, the space of continuous functions defined on $[0, 1]$ equipped with the max norm and $D = \bar{U}(0, 1)$. Define the operator F on D by*

$$F(x)(t) = x(t) - 5\int_0^1 t\theta x^3(\theta)d\theta,$$

so

$$F'(x)u(t) = u(t) - 15\int_0^1 t\theta x^2(\theta)u(\theta)d\theta \text{ for each } u \in D.$$

Then, we have for $p(t) = 0$ $(t \in [0, 1]))$, $\xi = 1$, that $L_0 = 7.5, L_3 = L = 15, w_0(t) = 7.5t, w_3(t) = w(t) = 15t$. Using (4.18), the radius in [19,33,39,41] is $\rho_1 = \frac{2}{3L_3} = 0.0444$ whereas the new radius is $\rho_3 = \frac{2}{2L_0+L3} = 0.0667$. Hence, our radius of convergence is larger.

5 Conclusion

The aim of this paper is to provide a finer local convergence analysis for Newton's method than in earlier papers using the gauge theory, the $w-$conditions, the center w_0-conditions and our new idea of restricted convergence regions. Using the center w_0-condition instead of the less precise $w-$condition used earlier, we find tighter upper bounds on the $\|F(x)^{-1}F(x_0)\|$. Moreover, using restricted convergence regions, we obtain a more precise location where the Newton iterates lie leading to tighter $w-$majorants. This way, we obtain a larger radius of convergence resulting to a wider choice of initial guesses and tighter error estimates on the distances $\|x_n - p\|$ leading to the calculation of fewer iterates to obtain a desired error tolerance. Moreover, the information on the location of the solution is also more accurate than in earlier studies. These, improvements are made under the same computational cost, since in practice the computation of old $w-$functions requires the computation of the new w_0- and w_1 functions as special cases.

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