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DYNAMICAL SYSTEM METHOD FOR ILL-POSED HAMMERSTEIN TYPE OPERATOR EQUATIONS WITH MONOTONE OPERATORS

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Abstract: The problem of approximately solving an ill-posed Hammerstein type operator equation $KF(x) = y$ in a Hilbert space is considered, where K is a bounded linear operator and F is a non-linear monotone operator.

The method involves the Dynamical System Method (DSM) – both continuous and iterative schemes, studied by Ramm (2005), and known as Tikhonov regularization. By choosing the regularization parameter according to an adaptive scheme considered by Pereverzev and Schock (2005) an order optimal error estimate has been obtained.

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Key Words: ill-posed Hammerstein type operator, dynamical system method, monotone operator, adaptive scheme, Tikhonov regularization

1. Introduction

In this paper we aim at obtaining an approximate solution for the nonlinear ill-posed(i.e., the solution does not depend continuously on the data) Hammerstein type ([4], [5], [6]) operator equation

$$KF(x) = f \tag{1}$$

using Dynamical System Method ([15]). Here $F : D(F) \subset X \to X$ is nonlinear

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monotone operator defined on a real Hilbert space X and $K : X \to Y$ is a bounded linear operator between the Hilbert spaces X and Y. The inner product and norm in X and Y are denoted by $\langle ., . \rangle$ and $\|.\|$ respectively. Recall that [11], [18], F is a monotone operator if $\langle F(x) - F(y), x - y \rangle \geq 0 \quad \forall x, y \in D(F)$.

In practice we only have noisy data f^δ with

$$\|f - f^\delta\| \leq \delta,$$

so one has to consider the equation

$$KF(x) = f^\delta \tag{2}$$

instead of (1).

Since (1) is ill-posed one has to consider regularization method to obtain a stable approximate solution for (1). Observe that the solution x of (2) can be obtained by first solving

$$Kz = f^\delta \tag{3}$$

for z and then solving the non-linear problem

$$F(x) = z \tag{4}$$

(See, [4], [5], [6], [7] and [8]). The advantage of approximately solving (3) and (4) to obtain an approximate solution for (2) is that, one can use any regularization method for linear ill-posed equations, for solving (3) and any method for non-linear ill-posed operator equations with monotone operators for solving (4).

As in [6] for approximately solving (3), we consider the Tikhonov regularized solution

$$z_\alpha^\delta = (K^*K + \alpha I)^{-1}K^*(f^\delta - KF(x_0)) + F(x_0), \;\; \alpha > 0, \delta > 0 \tag{5}$$

where x_0 is the known initial approximation to the solution $\hat{x}$ of (1).

Since F is monotone, for approximately solving (4) with z_α^δ in place of z we consider the Laverentiev regularization method, i.e., we consider the solution x_α^δ of the equation

$$F(x) + \frac{\alpha}{c}(x - x_0) = z_\alpha^\delta, \;\; c \leq \alpha \tag{6}$$

as an approximate solution of (4) with z_α^δ in place of z. In [15](Section 2.4.6, page 59), Ramm considered a method called Dynamical System Method (DSM) for solving

$$G(x) = 0. \tag{7}$$

The method consists of finding (cf. [13], [15]) a nonlinear locally Lipschitz operator $\Phi(u,t)$, such that the Cauchy problem:

$$u'(t) = \Phi(u,t), \qquad u(0) = u_0 \tag{8}$$

has the following three properties:

$$\exists u(t)\forall t \geq 0, \quad \exists u(\infty), \quad G(u(\infty)) = 0,$$

i.e., (8) is globally uniquely solvable, its unique solution has a limit at infinity $u(\infty)$, and this limit solves (7).

In [8] and [9], the authors considered a combination of modified form of DSM and Tikhonov regularization for obtaining a stable approximate solution of (2). The analysis in [8] and [9] was carried out under the assumption that, the Fréchet derivative $F'(x_0)$ of F at x_0 is invertible and is bounded. But in the present paper we analyze the case where $F'(x_0)^{-1}$ doesnot exist but F is a monotone operator.

We assume throughout that the solution $\hat{x}$ of (1) satisfies (See [8], [9])

$$\|\hat{x} - x_0\| = \min\{\|x - x_0\| : KF(x) = f, x \in D(F)\},$$

and that $\|\hat{x} - x_0\| \leq \rho$.

The regularization parameter α is chosen according to the adaptive scheme of Pereverzev and Schock ([14]). Here α is selected from some finite set $\alpha \in \{0 < \alpha_0 < \alpha_1 < \cdots < \alpha_N\}$ and the corresponding regularized solution, say $z^{\delta}_{\alpha_i}$, $0 \leq i \leq N$ are studied on-line.

The paper is organized as follows. In Section 2, we discuss the error bounds for Tikhonov regularization of (3) under general source conditions by choosing the regularization parameter by an a priori manner as well as by an adaptive scheme proposed by Pereverzev and Schock in [14]. In Section 3, continuous and iterative schemes of DSM are presented and we conclude the paper in Section 4.

2. Preparatory Results

The assumption below on source condition is based on a source function φ and a property of the source function φ. We will be using this assumption for error analysis.

Assumption 2.1. There exists a continuous, strictly monotonically increasing function $\varphi : (0, a] \to (0, \infty)$ with $a \geq \|K^*K\|$ satisfying;

- $\lim_{\lambda \to 0} \varphi(\lambda) = 0$
- $$\sup_{\lambda \geq 0} \frac{\alpha\varphi(\lambda)}{\lambda + \alpha} \leq \varphi(\alpha), \qquad \forall \lambda \in (0, a]$$
- there exists $v \in X$ such that
$$F(\hat{x}) = \varphi(K^*K)v.$$

Theorem 2.2. *Let z_α^δ be as in (5) and Assumption 2.1 hold. Then*

$$\|F(\hat{x}) - z_\alpha^\delta\| \leq \varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}. \tag{9}$$

Proof. See the proof of (4.3) in [6]. □

2.1. Error Bounds Under Source Conditions

Let $\alpha := \alpha_\delta$ be the minimum for the estimate $\varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}$ in Theorem 2.2. Then $\varphi(\alpha_\delta) = \frac{\delta}{\sqrt{\alpha_\delta}}$, i.e., $\delta = \sqrt{\alpha_\delta}\varphi(\alpha_\delta) = \psi(\varphi(\alpha_\delta))$, where $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}, 0 < \lambda \leq \|K\|^2$. So we have

$$\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta)), \tag{10}$$

and the relation (9) leads to $\|F(\hat{x}) - z_\alpha^\delta\| \leq 2\psi^{-1}(\delta)$.

2.2. An Adaptive Choice of the Parameter

From the above discussion the error estimate in Theorem 2.2 has optimal order with respect to δ for the choice of $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$. But the smoothness properties of the unknown solution $\hat{x}$ reflected in the function φ are generally not known, so in practice one cannot use the a priori parameter choice (10).

There exist many parameter choice strategies in the literature, for example see [2], [3], [7] and [18]. We employ the adaptive selection of the parameter suggested by Pereverzev and Schock in [14] which does not involve even the regularization method in an explicit manner.

Let $i \in \{0, 1, 2, \cdots, N\}$ and $\alpha_i = \mu^{2i}\alpha_0$ where $\mu > 1$. Let

$$l := max\{i : \varphi(\alpha_i) \leq \frac{\delta}{\sqrt{\alpha_i}}\} \tag{11}$$

and

$$k := max\{i : \|z^\delta_{\alpha_i} - z^\delta_{\alpha_j}\| \leq \frac{4\delta}{\sqrt{\alpha_j}}, j = 0, 1, 2, \cdots, i\}. \tag{12}$$

Theorem 2.3. (cf. [6], Theorem 4.3) *Let l be as in (11), k be as in (12) and $z^\delta_{\alpha_k}$ be as in (5) with $\alpha = \alpha_k$. Then $l \leq k$ and*

$$\|F(\hat{x}) - z^\delta_{\alpha_k}\| \leq (2 + \frac{4\mu}{\mu - 1})\mu\psi^{-1}(\delta).$$

3. Dynamical System Method (DSM)

The following Assumption is used throughout the analysis.

Assumption 3.1. (cf. [17], Assumption 3 (A3)) There exists a constant $k_0 \geq 0$ such that for every $x, u \in D(F)$ and $v \in X$ there exists an element $\Phi(x, u, v) \in X$ such that $[F'(x) - F'(u)]v = F'(u)\Phi(x, u, v), \|\Phi(x, u, v)\| \leq k_0\|v\|\|x - u\|$.

Throughout this section we assume that $F \in C^2$ i.e., $\forall x \in D(F)$,

$$\|F^{(j)}(x)\| \leq M_j, \qquad j = 1, 2. \tag{13}$$

Let $\delta_0 < \frac{2}{M_2+2k_0}\sqrt{\alpha_0}$ and

$$R_\rho := \frac{\delta_0}{\sqrt{\alpha_0}} + M\rho. \tag{14}$$

Lemma 3.2. *Let R_ρ be as in (14). Let $z^\delta_{\alpha_k}$ be as in (5), and if $x^\delta_{\alpha_k}$ is the solution of (6) with $\alpha := \alpha_k$ and $\delta \in [0, \delta_0]$, then $x^\delta_{\alpha_k} \in B_{R_\rho}(x_0)$.*

Proof. Observe that $F(x^\delta_{\alpha_k}) + \frac{\alpha_k}{c}(x^\delta_{\alpha_k} - x_0) = z^\delta_{\alpha_k}$. Let $M := \int_0^1 F'(x_0 + t(x^\delta_{\alpha_k} - x_0))dt$. Then

$$\begin{aligned} F(x^\delta_{\alpha_k}) - F(x_0) + \frac{\alpha_k}{c}(x^\delta_{\alpha_k} - x_0) &= z^\delta_{\alpha_k} - F(x_0) \\ (M + \frac{\alpha_k}{c}I)(x^\delta_{\alpha_k} - x_0) &= z^\delta_{\alpha_k} - F(x_0) \\ (x^\delta_{\alpha_k} - x_0) &= (M + \frac{\alpha_k}{c}I)^{-1}(z^\delta_{\alpha_k} - F(x_0)). \end{aligned}$$

Thus

$$\|x^\delta_{\alpha_k} - x_0\| \leq \|z^\delta_{\alpha_k} - F(x_0)\|$$

$$
\begin{aligned}
&\leq \|(K^*K+\alpha_k I)^{-1}K^*(f^\delta - KF(x_0))\| \\
&\leq \|(K^*K+\alpha_k I)^{-1}K^*(f^\delta - f + f - KF(x_0))\| \\
&\leq \|(K^*K+\alpha_k I)^{-1}K^*(f^\delta - f)\| \\
&\quad +\|(K^*K+\alpha_k I)^{-1}K^*K(F(\hat{x}) - F(x_0))\| \\
&\leq \frac{\delta}{\sqrt{\alpha_k}} + M\rho < R_\rho.
\end{aligned}
$$

Hence the Lemma. □

3.1. Continuous Schemes

In this section we consider the following Cauchy's problem for solving (4):

$$x'(t) = -(F'(x_0)+\frac{\alpha_k}{c}I)^{-1}(F(x(t))-z^\delta_{\alpha_k}+\frac{\alpha_k}{c}(x(t)-x_0)), \qquad x(0)=x_0 \quad (15)$$

where $c \leq \alpha_k$ and x_0 is an initial approximation. In this section we assume that

$$\rho < \frac{1}{M}[\frac{2}{M_2+2k_0} - \frac{\delta_0}{\sqrt{\alpha_0}}]. \quad (16)$$

Note that (16) implies that $R_\rho < \frac{1}{k_0}$.

The following Theorem gives the local solution for the Cauchy problem (15).

Theorem 3.3. *([13], Theorem 2.1) Let X be a real Banach space, U be an open subset of X and $x_0 \in U$. Let $\Phi : U \times \mathbb{R}^+ \to X$ be of class C^1 that is bounded on bounded sets. Then the following hold.*

- *There exists a maximal interval J containing 0 such that the initial value problem*

  $$x'(t) = \Phi(x(t), t), \qquad x(0) = x_0,$$

  has a unique solution $x(t) \in U$ for all $t \in J$.

- *If J has the right end point, say τ, and $x_\tau := \lim_{t\to\tau} x(t)$ exists, then x_τ is on the boundary of U.*

The Proposition below establishes the existence and uniqueness of the solution of the Cauchy problem (15).

Proposition 3.4. *Let F maps bounded sets onto bounded sets. Then there exists a maximal interval $J \subseteq [0,\infty)$ such that (15) has a unique solution $x(t)$ for all $t \in J$.*

Proof. Proof is analogous to the proof of Proposition 2.7 in [8]. □

Theorem 3.5. *Let $\delta \in [0, \delta_0]$, Assumption 3.1 and Lemma 3.2 be satisfied with ρ as in (16). If (13) and Proposition 3.4 hold, then (15) has a unique global solution $x(t)$ and $x(t)$ converges to $x^\delta_{\alpha_k}$. Further*

$$\|x(t) - x^\delta_{\alpha_k}\| \leq c_3 e^{-c_1 t}$$

where $c_3 = \frac{g(0)}{1-\frac{c_2 g(0)}{c_1}}$, $c_1 = 1 - k_0 R_\rho > 0$, $c_2 = \frac{M_2}{2}$ *and* $g(0) = \|x(0) - x^\delta_{\alpha_k}\|$.

Proof. Let $x(t) - x^\delta_{\alpha_k} := w$ and $\|w\| := g(t)$. Then by Taylor Theorem (cf. [1], Theorem 1.1.20)

$$F(x(t)) - F(x^\delta_{\alpha_k}) = F'(x^\delta_{\alpha_k})(x(t) - x^\delta_{\alpha_k}) + T(x(t), x^\delta_{\alpha_k}) \tag{17}$$

where $T(x(t), x^\delta_{\alpha_k}) = \int_0^1 F''(\lambda x(t) + (1-\lambda)x^\delta_{\alpha_k})(x(t) - x^\delta_{\alpha_k})^2(1-\lambda)d\lambda$. Since $F(x^\delta_{\alpha_k}) - z^\delta_{\alpha_k} + \frac{\alpha_k}{c}(x^\delta_{\alpha_k} - x_0) = 0$, by (17) we have

$$F(x(t)) - z^\delta_{\alpha_k} + \frac{\alpha_k}{c}(x(t) - x_0) = (F'(x^\delta_{\alpha_k}) + \frac{\alpha_k}{c}I)(x(t) - x^\delta_{\alpha_k}) + T(x(t), x^\delta_{\alpha_k}).$$

Observe that

$$w'(t) = x'(t) = -(F'(x_0) + \frac{\alpha_k}{c}I)^{-1}[(F'(x^\delta_{\alpha_k}) + \frac{\alpha_k}{c}I)(x(t) - x^\delta_{\alpha_k}) + T(x(t), x^\delta_{\alpha_k})]$$

and hence

$$\begin{aligned} gg' &= \frac{1}{2}\frac{dg^2}{dt} = \frac{1}{2}\frac{d}{dt}\langle w, w\rangle = \langle w, w'\rangle \\ &= \langle w, -(F'(x_0) + \frac{\alpha_k}{c}I)^{-1}[(F'(x^\delta_{\alpha_k}) + \frac{\alpha_k}{c}I)(x(t) - x^\delta_{\alpha_k}) + T(x(t), x^\delta_{\alpha_k})]\rangle \\ &= \langle w, -w\rangle + \langle w, \Lambda w\rangle + \langle w, -(F'(x_0) + \frac{\alpha_k}{c}I)^{-1}T(x(t), x^\delta_{\alpha_k})\rangle \\ &\leq -\|w\|^2 + \|\Lambda\|\|w\|^2 + \|(F'(x_0) + \frac{\alpha_k}{c}I)^{-1}T(x(t), x^\delta_{\alpha_k})\|\|w\| \\ &\leq -g^2 + \|\Lambda\|g^2 + \|T(x(t), x^\delta_{\alpha_k})\|g \end{aligned} \tag{18}$$

where $\Lambda = -(F'(x_0) + \frac{\alpha_k}{c}I)^{-1}(F'(x^\delta_{\alpha_k}) - F'(x_0))$. Note that

$$\begin{aligned} \|\Lambda\| &\leq \sup_{\|v\| \leq 1} \|(F'(x_0) + \frac{\alpha_k}{c}I)^{-1}[F'(x_0) - F'(x^\delta_{\alpha_k})]v\| \\ &\leq \sup_{\|v\| \leq 1} \|(F'(x_0) + \frac{\alpha_k}{c}I)^{-1}F'(x_0)\Phi(x^\delta_{\alpha_k}, x_0, v)\| \end{aligned}$$

$$\leq \quad k_0 R_\rho \|v\|, \tag{19}$$

the last step follows from Assumption 3.1. Again by (13),

$$\begin{aligned} \|(F'(x_0) + \frac{\alpha_k}{c} I)^{-1} T(x(t), x^\delta_{\alpha_k})\| &\leq \|T(x(t), x^\delta_{\alpha_k})\| \\ &\leq \frac{M_2 \|x(t) - x^\delta_{\alpha_k}\|^2}{2} \\ &\leq \frac{M_2 g^2}{2}. \end{aligned} \tag{20}$$

Therefore by (18), (19) and (20) we have

$$gg' \leq -g^2 + k_0 R_\rho g^2 + \frac{M_2}{2} g^3$$

and hence

$$g' \leq -c_1 g + c_2 g^2 \tag{21}$$

where $c_1 := 1 - k_0 R_\rho > 0$ and $c_2 := \frac{M_2}{2}$. So by solving (21) we get,

$$g(t) \leq c_3 e^{-c_1 t}.$$

□

Remark 3.6. Note that by Lemma 3.2, $g(0) = \|x_0 - x^\delta_{\alpha_k}\| \leq R_\rho$ and hence condition (16) implies $\frac{c_2 g(0)}{c_1} < 1$.

Assumption 3.7. There exists a continuous, strictly monotonically increasing function $\varphi_1 : (0, b] \to (0, \infty)$ with $b \geq \|F'(x_0)\|$ satisfying;

- $\lim_{\lambda \to 0} \varphi_1(\lambda) = 0$,
-
$$\sup_{\lambda \geq 0} \frac{\alpha \varphi_1(\lambda)}{\lambda + \alpha} \leq \varphi_1(\alpha) \qquad \forall \lambda \in (0, b]$$
 and
- there exists $v \in X$ with $\|v\| \leq 1$ (cf. [12]) such that
$$x_0 - \hat{x} = \varphi_1(F'(x_0))v.$$
- for each $x \in B_{R_\rho}(x_0)$ there exists a bounded linear operator $G(x, x_0)$ (cf. [16]) such that
$$F'(x) = F'(x_0) G(x, x_0)$$
 with $\|G(x, x_0)\| \leq k_1$.

Assume that $k_1 < \frac{1-k_0R_\rho}{1-c}$ and for the sake of simplicity assume that $\varphi_1(\alpha) \leq \varphi(\alpha)$ for $\alpha > 0$.

Theorem 3.8. (cf. [10], Theorem 3.7) *Suppose $x^\delta_{\alpha_k}$ is the solution of (6) with $\delta \in [0, \delta_0]$, and Assumptions 3.1 and 3.7 hold with ρ as in (16). Then*

$$\|\hat{x} - x^\delta_{\alpha_k}\| \leq \frac{\varphi_1(\alpha_k) + \|F(\hat{x}) - z^\delta_{\alpha_k}\|}{1-(1-c)k_1 - k_0R_\rho}.$$

In particular by Theorem 2.3,

$$\|\hat{x} - x^\delta_{\alpha_k}\| \leq \frac{\varphi_1(\alpha_k) + (2+\frac{4\mu}{\mu-1})\mu\psi^{-1}(\delta)}{1-(1-c)k_1 - k_0R_\rho}.$$

Proof. The proof is analogous to the proof of Theorem 3.7 in [10]. □

3.1.1. Error Analysis

The following Theorem is a consequence of Theorem 3.5 and Theorem 3.8.

Theorem 3.9. *Suppose (13), and assumptions in Theorem 3.5 and Theorem 3.8 hold with ρ as in (16), then*

$$\|\hat{x} - x(t)\| \leq \frac{\varphi_1(\alpha_k) + (2+\frac{4\mu}{\mu-1})\mu\psi^{-1}(\delta)}{1-(1-c)k_1 - k_0R_\rho} + c_3e^{-c_1t},$$

where c_1 and c_3 are as in Theorem 3.5.

Theorem 3.10. *Let $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}, 0 < \lambda \leq \|K\|^2$ and the assumptions of Theorem 3.9 are satisfied. Let*

$$T := min\{t : e^{-c_1t} < \frac{\delta}{\sqrt{\alpha_k}}\},$$

and $x(T)$ be the solution of the Cauchy's problem (15) with $z^\delta_{\alpha_k}$ in place of z^δ_α, with $\delta \in [0, \delta]$. Then

$$\|\hat{x} - x(T)\| = O(\psi^{-1}(\delta)).$$

3.2. Iterative Schemes

In this section we assume that $M_2 < 2$, $\delta_0 < \frac{2-M_2}{2k_0}\sqrt{\alpha_0}$ and

$$\rho < \frac{1}{M}[\frac{2-M_2}{2k_0} - \frac{\delta_0}{\sqrt{\alpha_0}}]. \tag{22}$$

Now we solve (4) with the following discretization scheme

$$x_{n+1} = x_n - h(F'(x_0) + \frac{\alpha_k}{c} I)^{-1}[F(x_n) - z^\delta_{\alpha_k} + \frac{\alpha_k}{c}(x_n - x_0)], h = constant > 0, \tag{23}$$

with $c \leq \alpha_k$. Let us consider the following Cauchy's problem:

$$w'_{n+1}(t) = -(F'(x_0) + \frac{\alpha_k}{c} I)^{-1}[F(w_{n+1}(t)) - z^\delta_{\alpha_k} + \frac{\alpha_k}{c}(w_{n+1}(t) - x_0)], \tag{24}$$

$w_{n+1}(t_n) = x_n,\ t_n \leq t \leq t_{n+1}$ where x_n is as in (23).

The existence and uniqueness of the solution of the Cauchy problem (24) can be established as in Proposition 3.4.

Theorem 3.11. *If $\delta \in [0, \delta_0]$, (13), Assumption 3.1 and Lemma 3.2 hold with ρ as in (22), then (24) has a unique global solution $w_{n+1}(t)$ and $w_{n+1}(t)$ converges to $x^\delta_{\alpha_k}$. Further*

$$\|w_{n+1}(t) - x^\delta_{\alpha_k}\| \leq \frac{e^{-\tilde{c_1} nh}}{1 - \frac{\tilde{c_0}}{\tilde{c_1}}} e^{-\tilde{c_1}(t - t_n)} \tag{25}$$

where $\tilde{c_0} = \frac{M_2}{2}$ and $\tilde{c_1} = 1 - k_0 R_\rho > 0$.

Proof. We shall prove (25) by induction. Clearly for $n = 0$ the result is true, suppose (25) is true for some n. Let $w_{n+1}(t) - x^\delta_{\alpha_k} := \tilde{w}$ and $\|\tilde{w}\| := \tilde{g}(t)$. Then by Taylor Theorem (cf. [1], Theorem 1.1.20)

$$\begin{aligned} F(w_{n+1}(t)) - z^\delta_{\alpha_k} + \frac{\alpha_k}{c}(w_{n+1}(t) - x_0) &= F(w_{n+1}(t)) - F(x^\delta_{\alpha_k}) \\ &\quad + \frac{\alpha_k}{c}(w_{n+1}(t) - x^\delta_{\alpha_k}) \\ &= F'(x^\delta_{\alpha_k})(w_{n+1}(t) - x^\delta_{\alpha_k}) \\ &\quad + T(w_{n+1}(t), x^\delta_{\alpha_k}) \\ &\quad + \frac{\alpha_k}{c}(w_{n+1}(t) - x^\delta_{\alpha_k}) \end{aligned} \tag{26}$$

where $T(w_{n+1}(t), x^\delta_{\alpha_k}) = \int_0^1 F''(\lambda w_{n+1}(t) + (1-\lambda)x^\delta_\alpha)(w_{n+1}(t) - x^\delta_{\alpha_k})^2(1-\lambda)d\lambda$. Observe that

$$\begin{aligned} \tilde{w}'(t) = w'_{n+1}(t) &= -(F'(x_0) + \frac{\alpha_k}{c} I)^{-1}[(F'(x^\delta_{\alpha_k}) + \frac{\alpha_k}{c} I)(w_{n+1}(t) - x^\delta_{\alpha_k}) \\ &\quad + T(w_{n+1}(t), x^\delta_{\alpha_k})] \end{aligned}$$

and hence

$$\tilde{g}\tilde{g}' = \frac{1}{2}\frac{d\tilde{g}^2}{dt} = \frac{1}{2}\frac{d}{dt}\langle \tilde{w}, \tilde{w} \rangle = \langle \tilde{w}, \tilde{w}' \rangle$$

$$
\begin{aligned}
&= \langle \tilde{w}, -(F'(x_0) + \frac{\alpha_k}{c} I)^{-1} [(F'(x_\alpha^\delta) + \frac{\alpha_k}{c} I)\tilde{w} + T(w_{n+1}(t), x_{\alpha_k}^\delta)] \rangle \\
&= \langle \tilde{w}, -\tilde{w} \rangle + \langle \tilde{w}, -(F'(x_0) + \frac{\alpha_k}{c} I)^{-1} T(w_{n+1}(t), x_{\alpha_k}^\delta) \rangle \\
&\quad + \langle \tilde{w}, -(F'(x_0) + \frac{\alpha_k}{c} I)^{-1} (F'(x_{\alpha_k}^\delta) - F'(x_0))\tilde{w} \rangle \qquad (27)
\end{aligned}
$$

Note that

$$
\begin{aligned}
\langle \tilde{w}, -(F'(x_0) + \frac{\alpha_k}{c} I)^{-1} [F'(x_{\alpha_k}^\delta) - F'(x_0)]\tilde{w} \rangle &\leq \|\tilde{w}\| \|(F'(x_0) + \frac{\alpha_k}{c} I)^{-1} \\
&\quad (F'(x_0) - F'(x_{\alpha_k}^\delta))\tilde{w}\| \\
&\leq \|\tilde{w}\| \|(F'(x_0) + \frac{\alpha_k}{c} I)^{-1} \\
&\quad F'(x_0)\Phi(x_{\alpha_k}^\delta, x_0, \tilde{w})\| \\
&\leq k_0 R_\rho \|\tilde{w}\|^2 \qquad (28)
\end{aligned}
$$

the last step follows from Assumption 3.1. Again by (26) and (13)

$$
\begin{aligned}
\langle \tilde{w}, -(F'(x_0) + \frac{\alpha_k}{c} I)^{-1} T(w_{n+1}(t), x_{\alpha_k}^\delta) \rangle &\leq \|\tilde{w}\| \|(F'(x_0) + \frac{\alpha_k}{c} I)^{-1} \\
&\quad T(w_{n+1}(t), x_{\alpha_k}^\delta)\| \\
&\leq \|\tilde{w}\| \|T(w_{n+1}(t), x_{\alpha_k}^\delta)\| \\
&\leq \|\tilde{w}\| \frac{M_2 \|x(t) - x_{\alpha_k}^\delta\|^2}{2} \\
&\leq \|\tilde{w}\| \frac{M_2 \tilde{g}^2}{2}. \qquad (29)
\end{aligned}
$$

Therefore by (27), (28) and (29) we have

$$\tilde{g}\tilde{g}' \leq -\tilde{g}^2 + k_0 R_\rho \tilde{g}^2 + \frac{M_2}{2}\tilde{g}^3$$

i.e.,

$$\tilde{g}' \leq -\tilde{c_1}\tilde{g} + \tilde{c_0}\tilde{g}^2,$$

and hence

$$\tilde{g}(t) \leq \tilde{r} e^{-\tilde{c_1}(t-t_n)}$$

where $\tilde{r} = \frac{\tilde{g}(t_n)}{1 - \frac{\tilde{c_0}\tilde{g}(t_n)}{\tilde{c_1}}}$. Note that $\tilde{r} = \frac{\tilde{g}(t_n)}{1 - \frac{\tilde{c_0}\tilde{g}(t_n)}{\tilde{c_1}}} \leq \frac{e^{-\tilde{c_1}nh}}{1 - \frac{\tilde{c_0}}{\tilde{c_1}}}$, condition (22) implies $\frac{\tilde{c_0}}{\tilde{c_1}} < 1$ and hence

$$\tilde{g}(t) \leq \frac{e^{-\tilde{c_1}nh}}{1 - \frac{\tilde{c_0}}{\tilde{c_1}}} e^{-\tilde{c_1}(t-t_n)}.$$

This completes the proof of the Theorem. □

Theorem 3.12. *Let* $w_{n+1}(t)$ *be the solution of (24) and* $z^\delta_{\alpha_k}$ *be as in (5) with* $\delta \in [0, \delta_0]$ *and* $\alpha = \alpha_k$. *If Lemma 3.2 holds with* ρ *as in (22), then*

$$\|F(w_{n+1}(t)) - z^\delta_{\alpha_k} + \frac{\alpha_k}{c}(w_{n+1}(t) - x_0)\| \leq \|F(x_0) - z^\delta_{\alpha_k}\| e^{-\tilde{c_1}(nh+t-t_n)}. \quad (30)$$

Proof. The proof follows as in proof of Theorem 3.11 by taking

$$\tilde{g}(t) = \|F(w_{n+1}(t)) - z^\delta_{\alpha_k} + \frac{\alpha_k}{c}(w_{n+1}(t) - x_0)\|.$$

□

Proposition 3.13. *Let* x_{n+1} *be as in (23) with* $\delta \in [0, \delta_0]$. *If (13) and Theorem 3.12 hold, then*

$$\|x_{n+1} - w_{n+1}(t_{n+1})\| \leq h^2(M_1 + 1)R_\rho e^{-\tilde{c_1}nh}.$$

Proof. Observe that

$$\begin{aligned}
\|x_{n+1} - w_{n+1}(t_{n+1})\| &= \int_{t_n}^{t_{n+1}} \|\Phi(x_n) - \Phi(w_{n+1}(t))\| dt \\
&\leq \int_{t_n}^{t_{n+1}} \|(F'(x_0) + \frac{\alpha_k}{c}I)^{-1}[F(x_n) - F(w_{n+1}(t)) \\
&\quad + \frac{\alpha_k}{c}(x_n - w_{n+1}(t))]\| dt \\
&\leq (M_1 + 1)\int_{t_n}^{t_{n+1}} \|x_n - w_{n+1}(t)\| dt \\
&\leq (M_1 + 1)h \int_{t_n}^{t_{n+1}} \|\Phi(w_{n+1}(t))\| dt \\
&\leq (M_1 + 1)h \int_{t_n}^{t_{n+1}} \|(F'(x_0) + \frac{\alpha_k}{c}I)^{-1}[F(w_{n+1}(t)) - z^\delta_{\alpha_k} \\
&\quad + \frac{\alpha_k}{c}(w_{n+1}(t) - x_0)]\| dt. \qquad (31)
\end{aligned}$$

Now from (30), (31) and Lemma 3.2 we have,

$$\begin{aligned}
\|x_{n+1} - w_{n+1}(t_{n+1})\| &\leq h^2(M_1 + 1)\|F(x_0) - z^\delta_{\alpha_k}\| e^{-\tilde{c_1}nh} \\
&\leq h^2(M_1 + 1)R_\rho e^{-\tilde{c_1}nh}.
\end{aligned}$$

Hence the Proposition. □

Thus by triangle inequality, (25) and (31) we have the following

Theorem 3.14. *If the assumptions of Proposition 3.13 and Theorem 3.11 hold. Then* x_{n+1} *converges to* $x_{\alpha_k}^\delta$. *Further*

$$\|x_{n+1} - x_{\alpha_k}^\delta\| \le \tilde{C}e^{-\tilde{c_1}nh}$$

where $\tilde{C} = h^2(M_1+1)R_\rho + \frac{1}{1-\frac{\tilde{c_0}}{\tilde{c_1}}}e^{-\tilde{c_1}h}$.

3.2.1. Error Analysis

Theorem 3.15. *Let assumptions of Theorem 3.14 hold. Suppose* $k_1 < \frac{1-k_0R_\rho}{1-c}$ *and assumptions of Theorem 3.8 hold with* ρ *as in (22), then*

$$\|\hat{x} - x_{n+1}\| \le \frac{\varphi_1(\alpha_k) + (2+\frac{4\mu}{\mu-1})\mu\psi^{-1}(\delta)}{1-(1-c)k_1 - k_0R_\rho} + \tilde{C}e^{-\tilde{c_1}nh}.$$

Proof. The proof follows from Theorem 3.14, Theorem 3.8(with ρ as in (22)) and the triangle inequality:

$$\|\hat{x} - x_{n+1}\| \le \|\hat{x} - x_{\alpha_k}^\delta\| + \|x_{\alpha_k}^\delta - x_{n+1}\|.$$

□

Theorem 3.16. *Let* $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}, 0 < \lambda \le \|K\|^2$ *and the assumptions of Theorem 3.15 are satisfied. Let*

$$N := min\{n : e^{-\tilde{c_1}nh} < \frac{\delta}{\sqrt{\alpha_\delta}}\}$$

and x_{N+1} *be as in (23) with* $z_{\alpha_k}^\delta$ *in place of* z_α^δ, *with* $\delta \in [0,\delta]$. *Then*

$$\|\hat{x} - x_{N+1}\| = O(\psi^{-1}(\delta)).$$

4. Conclusion

In this paper we presented a method, which is a combination of DSM and Tikhonov regularization method for approximately solving ill-posed Hammerstein type operator equation $KF(x) = f$, when the available data is f^δ with $\|f - f^\delta\|$ and the non-linear operator F is monotone. Infact we considered continuous and iterative schemes of DSM studied extensively by Ramm (see [15]) and his collaborators. We obtained order optimal error bounds by choosing the regularization parameter α according to the adaptive method considered by Pereverzev and Schock(2005). Further in a future work it is envisaged to investigate the case when F is non-invertible and non-monotone operator.

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