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Analytical solutions using a higher order refined computational model with 12 degrees of freedom for the free vibration analysis of antisymmetric angle-ply plates

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Abstract

Analytical formulations and solutions to the natural frequency analysis of simply supported antisymmetric angle-ply composite and sandwich plates hitherto not reported in the literature based on a higher order refined computational model with 12 degrees of freedom already reported in the literature are presented. The theoretical model presented herein incorporates laminate deformations which account for the effects of transverse shear deformation, transverse normal strain/stress and a nonlinear variation of in-plane displacements with respect to the thickness coordinate thus modelling the warping of transverse cross sections more accurately and eliminating the need for shear correction coefficients. In addition, another higher order computational model with five degrees of freedom already available in the literature is also considered for comparison. The equations of motion are obtained using Hamilton's principle. Solutions are obtained in closed-form using Navier's technique by solving the eigenvalue equation. Plates with varying slenderness ratios, number of layers, degrees of anisotropy, edge ratios and thickness of core to thickness of face sheet ratios are considered for analysis. Numerical results with real properties using above two computational models are presented and compared for the free vibration analysis of multilayer antisymmetric angle-ply composite and sandwich plates, which will serve as a benchmark for future investigations.

Keywords: Free vibration; Higher order theory; Shear deformation; Angle-ply plates; Analytical solutions

1. Introduction

Laminated composite and sandwich plates and shells are finding extensive usage in the aeronautical and aerospace industries as well as in other fields of modern technology. It has been observed that the strength and deformation characteristics of such structural elements depend upon the fibre orientation, stacking sequence and the fibre content in addition to the strength and rigidities of the fibre and matrix material. Though symmetric laminates are simple to analyse and design, some specific application of composite and sandwich laminates requires the use of unsymmetric laminates to fulfil certain design requirements. Antisymmetric cross-ply and angle-ply laminates are the special form of unsymmetric laminates and the associated theory offers some simplification in the analysis. The Classical Laminate Plate Theory [1] which ignores the effect of transverse shear deformation becomes inadequate for the analysis of multilayer composites. The First Order Shear Deformation Theories (FSDTs) based on Reissner [2] and Mindlin [3] assume linear in-plane stresses and displacements respectively through the laminate thickness. Since FSDTs account for layerwise constant states of transverse shear stress, shear correction coefficients are needed to rectify the unrealistic variation of the shear strain/stress through the thickness. In order to overcome the limitations of FSDTs, higher order shear deformation theories (HSDTs) that involve higher order terms in the Taylor's expansions of the displacement in the thickness coordinate were developed. Hildebrand et al. [4] were the first to introduce this approach to derive improved theories of plates and shells. Using the higher order theory of Reddy [5] free vibration analysis of isotropic, orthotropic and laminated

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plates was carried out by Reddy and Phan [6]. A selective review of the various analytical and numerical methods used for the stress analysis of laminated composite and sandwich plates was presented by Kant and Swaminathan [7]. Using the higher order refined theories already reported in the literature by Kant [8], Pandya and Kant [9–13] and Kant and Manjunatha [14], analytical formulations, solutions and comparison of numerical results for the buckling, free vibration and stress analyses of cross-ply composite and sandwich plates were presented by Kant and Swaminathan [15–18] and the finite element formulations and solutions for the free vibration analysis of multilayer plates were presented by Mallikarjuna [19], Mallikarjuna and Kant [20], Kant and Mallikarjuna [21,22]. Recently the theoretical formulations and solutions for the static analysis of antisymmetric angle-ply laminated composite and sandwich plates using various higher order refined computational models were presented by Swaminathan and Ragounadin [23], Swaminathan et al. [24] and Swaminathan and Patil [25]. In this paper, analytical formulations developed and solutions obtained for the first time using a higher order refined computational model with 12 degrees of freedom is presented for the free vibration analysis of antisymmetric angle-ply laminated composite and sandwich plates. In addition, another higher order model with five degrees of freedom already reported in the literature is also considered for the analysis. Results generated using both the models are presented for the antisymmetric angle-ply composite and sandwich plates with real properties.

2. Theoretical formulation

2.1. Displacement model

In order to approximate the three-dimensional elasticity problem to a two-dimensional plate problem, the displacement components $u(x,y,z,t), v(x,y,z,t)$ and $w(x,y,z,t)$ at any point in the plate space are expanded in Taylor's series in terms of the thickness coordinate. The elasticity solution indicates that the transverse shear stresses vary parabolically through the plate thickness. This requires the use of a displacement field in which the in-plane displacements are expanded as cubic functions of the thickness coordinate. In addition, the transverse normal strain may vary nonlinearly through the plate thickness. The displacement field which satisfies the above criteria may be assumed in the form [14]:

$$\begin{aligned} u(x,y,z,t) &= u_o(x,y,t) + z\theta_x(x,y,t) \\ &\quad + z^2 u_o^*(x,y,t) + z^3\theta_x^*(x,y,t) \\ v(x,y,z,t) &= v_o(x,y,t) + z\theta_y(x,y,t) \\ &\quad + z^2 v_o^*(x,y,t) + z^3\theta_y^*(x,y,t) \\ w(x,y,z,t) &= w_o(x,y,t) + z\theta_z(x,y,t) \\ &\quad + z^2 w_o^*(x,y,t) + z^3\theta_z^*(x,y,t) \end{aligned} \tag{1}$$

The parameters u_o, v_o are the in-plane displacements and w_o is the transverse displacement of a point (x,y) on the middle plane. The functions θ_x, θ_y are rotations of the normal to the middle plane about y and x axes respectively. The parameters $u_o^*, v_o^*, w_o^*, \theta_x^*, \theta_y^*, \theta_z^*$ and θ_z are the higher order terms in the Taylor's series expansion and they represent higher order transverse cross sectional deformation modes. Though the above theory was already reported earlier in the literature and numerical results were presented using finite element formulations, analytical formulations and solutions are obtained for the first time in this investigation and hence the results obtained using the above theory are referred to as *present* in all the tables. In addition to the above, the following higher order shear deformation theory [HSDT] with five degrees of freedom already reported in the literature for the analysis of laminated composite and sandwich plates are also considered for the evaluation purpose. Results using these theories are generated independently and presented here with a view to have all the results on a common platform.

Reddy [5]

$$\begin{aligned} u(x,y,z,t) &= u_o(x,y,t) + z\left[\theta_x(x,y,t) \right. \\ &\quad \left. - \frac{4}{3}\left(\frac{z}{h}\right)^2\left\{\theta_x(x,y,t) + \frac{\partial w_o}{\partial x}\right\}\right] \\ v(x,y,z,t) &= v_o(x,y,t) + z\left[\theta_y(x,y,t) \right. \\ &\quad \left. - \frac{4}{3}\left(\frac{z}{h}\right)^2\left\{\theta_y(x,y,t) + \frac{\partial w_o}{\partial y}\right\}\right] \\ w(x,y,z,t) &= w_o(x,y,t) \end{aligned} \tag{2}$$

In this paper the analytical formulations and solution method followed using the higher order refined theory given by Eq. (1) is presented in detail. The geometry of a two-dimensional laminated composite and sandwich plates with positive set of coordinate axes and the physical middle plane displacement terms are shown in Figs. 1 and 2 respectively. By substitution of the displacement relations given by Eq. (1) into the strain–displacement equations of the classical theory of elasticity, the following relations are obtained.

$$\begin{aligned} \varepsilon_x &= \varepsilon_{xo} + z\kappa_x + z^2\varepsilon_{xo}^* + z^3\kappa_x^* \\ \varepsilon_y &= \varepsilon_{yo} + z\kappa_y + z^2\varepsilon_{yo}^* + z^3\kappa_y^* \\ \varepsilon_z &= \varepsilon_{zo} + z\kappa_z^* + z^2\varepsilon_{zo}^* \\ \gamma_{xy} &= \varepsilon_{xyo} + z\kappa_{xy} + z^2\varepsilon_{xyo}^* + z^3\kappa_{xy}^* \\ \gamma_{yz} &= \phi_y + z\kappa_{yz} + z^2\phi_y^* + z^3\kappa_{yz}^* \\ \gamma_{xz} &= \phi_x + z\kappa_{xz} + z^2\phi_x^* + z^3\kappa_{xz}^* \end{aligned} \tag{3}$$

where

$$(\varepsilon_{xo}, \varepsilon_{yo}, \varepsilon_{xyo}) = \left(\frac{\partial u_o}{\partial x}, \frac{\partial v_o}{\partial y}, \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right)$$

$$(\varepsilon_{xo}^*, \varepsilon_{yo}^*, \varepsilon_{xyo}^*) = \left(\frac{\partial u_o^*}{\partial x}, \frac{\partial v_o^*}{\partial y}, \frac{\partial u_o^*}{\partial y} + \frac{\partial v_o^*}{\partial x}\right)$$

$$(\varepsilon_{zo}, \varepsilon_{zo}^*) = (\theta_z, 3\theta_z^*)$$

$$(\kappa_x, \kappa_y, \kappa_z^*, \kappa_{xy}) = \left(\frac{\partial \theta_x}{\partial x}, \frac{\partial \theta_y}{\partial y}, 2w_o^*, \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\right)$$

$$(\kappa_x^*, \kappa_y^*, \kappa_{xy}^*) = \left(\frac{\partial \theta_x^*}{\partial x}, \frac{\partial \theta_y^*}{\partial y}, \frac{\partial \theta_x^*}{\partial y} + \frac{\partial \theta_y^*}{\partial x}\right)$$

$$(\kappa_{xz}, \kappa_{yz}) = \left(2u_o^* + \frac{\partial \theta_z}{\partial x}, 2v_o^* + \frac{\partial \theta_z}{\partial y}\right)$$

$$(\kappa_{xz}^*, \kappa_{yz}^*) = \left(\frac{\partial \theta_z^*}{\partial x}, \frac{\partial \theta_z^*}{\partial y}\right)$$

$$(\phi_x, \phi_x^*, \phi_y, \phi_y^*) = \left(\theta_x + \frac{\partial w_o}{\partial x}, 3\theta_x^* + \frac{\partial w_o^*}{\partial x}, \theta_y + \frac{\partial w_o}{\partial y}, 3\theta_y^* + \frac{\partial w_o^*}{\partial y}\right) \quad (4)$$

2.2. Constitutive equations

Each lamina in the laminate is assumed to be in a three-dimensional stress state so that the constitutive relation for a typical lamina L with reference to the fibre–matrix coordinate axes (1–2–3) can be written as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{12} \\ \tau_{23} \\ \tau_{13} \end{Bmatrix}^L = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}^L \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{Bmatrix}^L \quad (5)$$

where $(\sigma_1, \sigma_2, \sigma_3, \tau_{12}, \tau_{23}, \tau_{13})$ are the stresses and $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \gamma_{12}, \gamma_{23}, \gamma_{13})$ are the linear strain components referred to the lamina coordinates (1–2–3) and the C_{ij}'s are the elastic constants or the elements of stiffness matrix [25] of the Lth lamina with reference to the fibre axes (1–2–3). In the laminate coordinate (x, y, z) the stress strain relations for the Lth lamina can be written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix}^L = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} & 0 & 0 \\ & Q_{22} & Q_{23} & Q_{24} & 0 & 0 \\ & & Q_{33} & Q_{34} & 0 & 0 \\ & & & Q_{44} & 0 & 0 \\ & \text{symmetric} & & & & \\ & & & & Q_{55} & Q_{56} \\ & & & & & Q_{66} \end{bmatrix}^L \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}^L \quad (6)$$

where $(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{xz})$ are the stresses and $(\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{xz})$ are the strains with respect to the laminate axes. Q_{ij}'s are the transformed elastic constants or the stiffness matrix [25] with respect to the laminate axes x, y, z.

2.3. Hamilton's principle

Hamilton's principle [26] can be written in analytical form as follows:

$$\delta \int_{t_1}^{t_2} [K - (U + V)]\, \mathrm{d}t = 0 \quad (7)$$

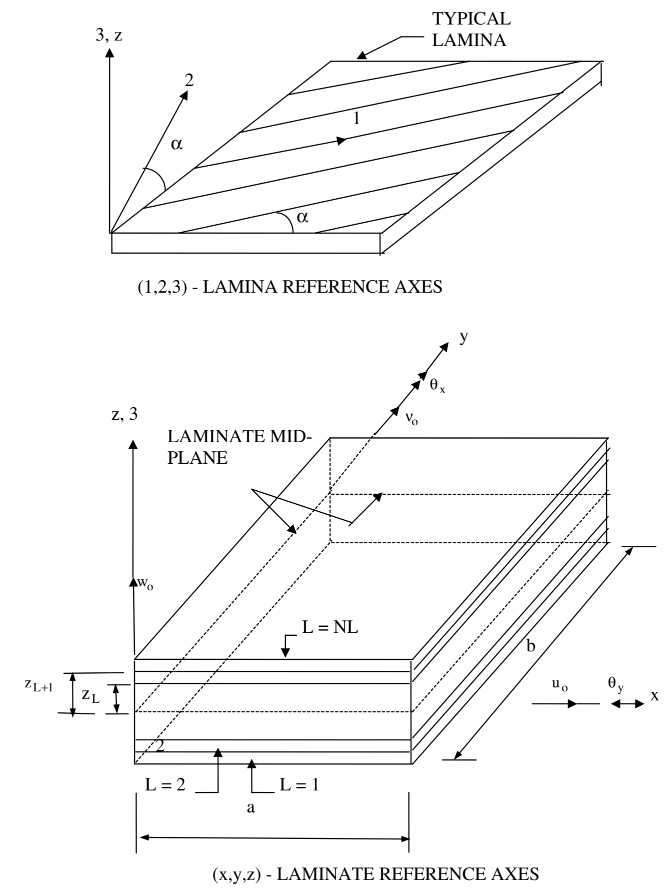

Fig. 1. Laminate geometry with positive set of lamina/laminate reference axes, displacement components and fibre orientation.

where U is the total strain energy due to deformations, V is the potential of the external loads, K is the kinetic energy and $U + V = \Pi$ is the total potential energy and δ denotes the variational symbol. Substituting the appropriate energy expression in the above equation, the final expression can thus be written as

$$0 = -\int_0^t \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \int_A (\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{xz}\delta\gamma_{xz})\, \mathrm{d}A\, \mathrm{d}z - \int_A p_z^+ \delta w^+ \, \mathrm{d}A\right] \mathrm{d}t + \frac{\delta}{2}\int_0^t \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_A \rho[(\dot{u})^2 + (\dot{v})^2 + (\dot{w})^2]\, \mathrm{d}A\, \mathrm{d}z\, \mathrm{d}t \quad (8)$$

where ρ is the mass density of the material of the laminate and p_z^+ is the transverse load applied at the top surface of the plate and $w^+ = w_o + (h/2)\theta_z + (h^2/4)w_o^* + (h^3/8)\theta_z^*$ is the transverse displacement of any point on the top surface of the plate and the superposed dot denotes differentiation with respect to time. Using Eqs. (1), (3) and (4) in Eq. (8) and integrating the resulting expression by parts, and collecting the coefficients of $\delta u_0, \delta v_0, \delta w_0, \delta\theta_x, \delta\theta_y, \delta\theta_z, \delta u_0^*, \delta v_0^*, \delta w_0^*, \delta\theta_x^*, \delta\theta_y^*, \delta\theta_z^*$ the following equations of equilibrium are obtained:

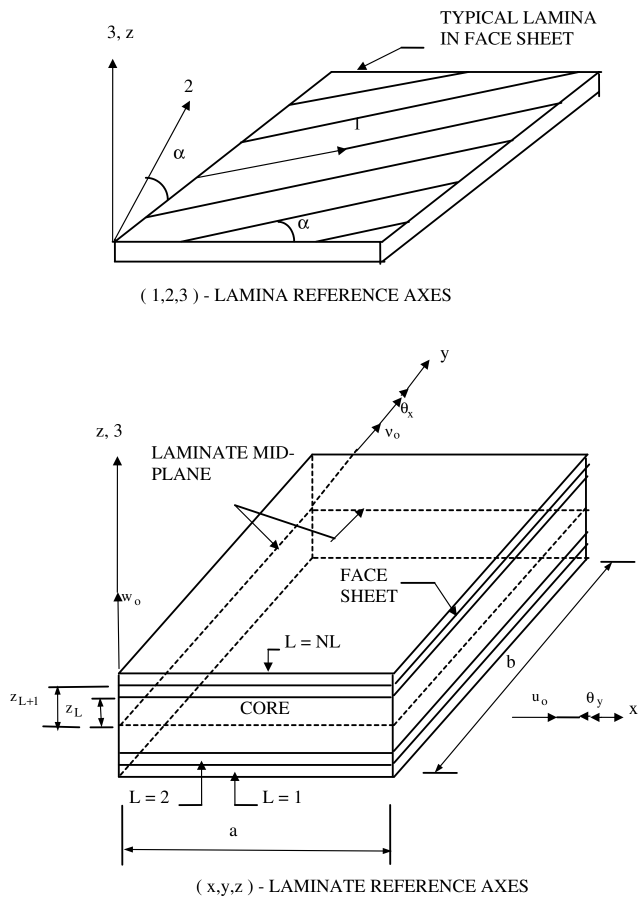

Fig. 2. Geometry of a sandwich plate with positive set of lamina/laminate reference axes, displacement components and fibre orientation.

$$\delta u_o : \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_1 \ddot{u}_o + I_2 \ddot{\theta}_x + I_3 \ddot{u}_o^* + I_4 \ddot{\theta}_x^*$$

$$\delta v_o : \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = I_1 \ddot{v}_o + I_2 \ddot{\theta}_y + I_3 \ddot{v}_o^* + I_4 \ddot{\theta}_y^*$$

$$\delta w_o : \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p_z^+ = I_1 \ddot{w}_o + I_2 \ddot{\theta}_z + I_3 \ddot{w}_o^* + I_4 \ddot{\theta}_z^*$$

$$\delta \theta_x : \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = I_2 \ddot{u}_o + I_3 \ddot{\theta}_x + I_4 \ddot{u}_o^* + I_5 \ddot{\theta}_x^*$$

$$\delta \theta_y : \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = I_2 \ddot{v}_o + I_3 \ddot{\theta}_y + I_4 \ddot{v}_o^* + I_5 \ddot{\theta}_y^*$$

$$\delta \theta_z : \frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} - N_z + \frac{h}{2}\left(p_z^+\right) = I_2 \ddot{w}_o + I_3 \ddot{\theta}_z + I_4 \ddot{w}_o^* + I_5 \ddot{\theta}_z^*$$

$$\delta u_o^* : \frac{\partial N_x^*}{\partial x} + \frac{\partial N_{xy}^*}{\partial y} - 2S_x = I_3 \ddot{u}_o + I_4 \ddot{\theta}_x + I_5 \ddot{u}_o^* + I_6 \ddot{\theta}_x^*$$

$$\delta v_o^* : \frac{\partial N_y^*}{\partial y} + \frac{\partial N_{xy}^*}{\partial x} - 2S_y = I_3 \ddot{v}_o + I_4 \ddot{\theta}_y + I_5 \ddot{v}_o^* + I_6 \ddot{\theta}_y^*$$

$$\delta w_o^* : \frac{\partial Q_x^*}{\partial x} + \frac{\partial Q_y^*}{\partial y} - 2M_z^* + \frac{h^2}{4}\left(p_z^+\right) = I_3 \ddot{w}_o + I_4 \ddot{\theta}_z + I_5 \ddot{w}_o^* + I_6 \ddot{\theta}_z^*$$

$$\delta \theta_x^* : \frac{\partial M_x^*}{\partial x} + \frac{\partial M_{xy}^*}{\partial y} - 3Q_x^* = I_4 \ddot{u}_o + I_5 \ddot{\theta}_x + I_6 \ddot{u}_o^* + I_7 \ddot{\theta}_x^*$$

$$\delta \theta_y^* : \frac{\partial M_y^*}{\partial y} + \frac{\partial M_{xy}^*}{\partial x} - 3Q_y^* = I_4 \ddot{v}_o + I_5 \ddot{\theta}_y + I_6 \ddot{v}_o^* + I_7 \ddot{\theta}_y^*$$

$$\delta \theta_z^* : \frac{\partial S_x^*}{\partial x} + \frac{\partial S_y^*}{\partial y} - 3N_z^* + \frac{h^3}{8}\left(p_z^+\right) = I_4 \ddot{w}_o + I_5 \ddot{\theta}_z + I_6 \ddot{w}_o^* + I_7 \ddot{\theta}_z^* \quad (9)$$

and boundary conditions are the form:

On the edge $x = \text{constant}$

$$\begin{aligned}
&u_o = \overline{u}_o \quad \text{or} \quad N_x = \overline{N}_x \qquad u_o^* = \overline{u}_o^* \quad \text{or} \quad N_x^* = \overline{N}_x^* \\
&v_o = \overline{v}_o \quad \text{or} \quad N_{xy} = \overline{N}_{xy} \qquad v_o^* = \overline{v}_o^* \quad \text{or} \quad N_{xy}^* = \overline{N}_{xy}^* \\
&w_o = \overline{w}_o \quad \text{or} \quad Q_x = \overline{Q}_x \qquad w_o^* = \overline{w}_o^* \quad \text{or} \quad Q_x^* = \overline{Q}_x^* \\
&\theta_x = \overline{\theta}_x \quad \text{or} \quad M_x = \overline{M}_x \qquad \theta_x^* = \overline{\theta}_x^* \quad \text{or} \quad M_x^* = \overline{M}_x^* \\
&\theta_y = \overline{\theta}_y \quad \text{or} \quad M_{xy} = \overline{M}_{xy} \qquad \theta_y^* = \overline{\theta}_y^* \quad \text{or} \quad M_{xy}^* = \overline{M}_{xy}^* \\
&\theta_z = \overline{\theta}_z \quad \text{or} \quad S_x = \overline{S}_x \qquad \theta_z^* = \overline{\theta}_z^* \quad \text{or} \quad S_x^* = \overline{S}_x^*
\end{aligned} \quad (10)$$

On the edge $y = \text{constant}$

$$\begin{aligned}
&u_o = \overline{u}_o \quad \text{or} \quad N_{xy} = \overline{N}_{xy} \qquad u_o^* = \overline{u}_o^* \quad \text{or} \quad N_{xy}^* = \overline{N}_{xy}^* \\
&v_o = \overline{v}_o \quad \text{or} \quad N_y = \overline{N}_y \qquad v_o^* = \overline{v}_o^* \quad \text{or} \quad N_y^* = \overline{N}_y^* \\
&w_o = \overline{w}_o \quad \text{or} \quad Q_y = \overline{Q}_y \qquad w_o^* = \overline{w}_o^* \quad \text{or} \quad Q_y^* = \overline{Q}_y^* \\
&\theta_x = \overline{\theta}_x \quad \text{or} \quad M_{xy} = \overline{M}_{xy} \qquad \theta_x^* = \overline{\theta}_x^* \quad \text{or} \quad M_{xy}^* = \overline{M}_{xy}^* \\
&\theta_y = \overline{\theta}_y \quad \text{or} \quad M_y = \overline{M}_y \qquad \theta_y^* = \overline{\theta}_y^* \quad \text{or} \quad M_y^* = \overline{M}_y^* \\
&\theta_z = \overline{\theta}_z \quad \text{or} \quad S_y = \overline{S}_y \qquad \theta_z^* = \overline{\theta}_z^* \quad \text{or} \quad S_y^* = \overline{S}_y^*
\end{aligned} \quad (11)$$

where the stress resultants are defined by

$$\begin{bmatrix} M_x & M_x^* \\ M_y & M_y^* \\ M_z^* & 0 \\ M_{xy} & M_{xy}^* \end{bmatrix} = \sum_{L=1}^{NL} \int_{z_L}^{z_{L+1}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \end{Bmatrix} \begin{bmatrix} z & z^3 \end{bmatrix} \mathrm{d}z \quad (12)$$

$$\begin{bmatrix} Q_x & Q_x^* \\ Q_y & Q_y^* \end{bmatrix} = \sum_{L=1}^{NL} \int_{z_L}^{z_{L+1}} \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} \begin{bmatrix} 1 & z^2 \end{bmatrix} \mathrm{d}z \quad (13)$$

$$\begin{bmatrix} N_x & N_x^* \\ N_y & N_y^* \\ N_z & N_z^* \\ N_{xy} & N_{xy}^* \end{bmatrix} = \sum_{L=1}^{NL} \int_{z_L}^{z_{L+1}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \end{Bmatrix} \begin{bmatrix} 1 & z^2 \end{bmatrix} \mathrm{d}z \quad (14)$$

$$\begin{bmatrix} S_x & S_x^* \\ S_y & S_y^* \end{bmatrix} = \sum_{L=1}^{NL} \int_{z_L}^{z_{L+1}} \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} \begin{bmatrix} z & z^3 \end{bmatrix} \mathrm{d}z \quad (15)$$

and the inertias are given by

$$I_1, I_2, I_3, I_4, I_5, I_6, I_7, = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(1, z, z^2, z^3, z^4, z^5, z^6,)\, \mathrm{d}z \quad (16)$$

The resultants in Eqs. (12)–(15) can be related to the total strains in Eq. (3) by the following equations:

$$\begin{Bmatrix} N_x \\ N_y \\ N_x^* \\ N_y^* \\ N_z \\ N_z^* \\ M_x \\ M_y \\ M_x^* \\ M_y^* \\ M_z^* \end{Bmatrix} = [A] \begin{Bmatrix} \frac{\partial u_o}{\partial x} \\ \frac{\partial v_o}{\partial y} \\ \frac{\partial u_o^*}{\partial x} \\ \frac{\partial v_o^*}{\partial y} \\ \theta_z \\ \theta_z^* \\ \frac{\partial \theta_x}{\partial x} \\ \frac{\partial \theta_y}{\partial y} \\ \frac{\partial \theta_x^*}{\partial x} \\ \frac{\partial \theta_y^*}{\partial y} \\ w_o^* \end{Bmatrix} + [A'] \begin{Bmatrix} \frac{\partial u_o}{\partial y} \\ \frac{\partial v_o}{\partial x} \\ \frac{\partial u_o^*}{\partial y} \\ \frac{\partial v_o^*}{\partial x} \\ \frac{\partial \theta_x}{\partial y} \\ \frac{\partial \theta_y}{\partial x} \\ \frac{\partial \theta_x^*}{\partial y} \\ \frac{\partial \theta_y^*}{\partial x} \end{Bmatrix} \tag{17}$$

$$\begin{Bmatrix} N_{xy} \\ N_{xy}^* \\ M_{xy} \\ M_{xy}^* \end{Bmatrix} = [B'] \begin{Bmatrix} \frac{\partial u_o}{\partial x} \\ \frac{\partial v_o}{\partial y} \\ \frac{\partial u_o^*}{\partial x} \\ \frac{\partial v_o^*}{\partial y} \\ \theta_z \\ \theta_z^* \\ \frac{\partial \theta_x}{\partial x} \\ \frac{\partial \theta_y}{\partial y} \\ \frac{\partial \theta_x^*}{\partial x} \\ \frac{\partial \theta_y^*}{\partial y} \\ w_o^* \end{Bmatrix} + [B] \begin{Bmatrix} \frac{\partial u_o}{\partial y} \\ \frac{\partial v_o}{\partial x} \\ \frac{\partial u_o^*}{\partial y} \\ \frac{\partial v_o^*}{\partial x} \\ \frac{\partial \theta_x}{\partial y} \\ \frac{\partial \theta_y}{\partial x} \\ \frac{\partial \theta_x^*}{\partial y} \\ \frac{\partial \theta_y^*}{\partial x} \end{Bmatrix}$$

$$\begin{Bmatrix} Q_x \\ Q_x^* \\ S_x \\ S_x^* \end{Bmatrix} = [D] \begin{Bmatrix} \theta_x \\ \frac{\partial w_o}{\partial x} \\ \theta_x^* \\ \frac{\partial w_o^*}{\partial x} \\ u_o^* \\ \frac{\partial \theta_z}{\partial x} \\ \frac{\partial \theta_z^*}{\partial x} \end{Bmatrix} + [D'] \begin{Bmatrix} \theta_y \\ \frac{\partial w_o}{\partial y} \\ \theta_y^* \\ \frac{\partial w_o^*}{\partial y} \\ v_o^* \\ \frac{\partial \theta_z}{\partial y} \\ \frac{\partial \theta_z^*}{\partial y} \end{Bmatrix} \tag{18}$$

$$\begin{Bmatrix} Q_y \\ Q_y^* \\ S_y \\ S_y^* \end{Bmatrix} = [E'] \begin{Bmatrix} \theta_x \\ \frac{\partial w_o}{\partial x} \\ \theta_x^* \\ \frac{\partial w_o^*}{\partial x} \\ u_o^* \\ \frac{\partial \theta_z}{\partial x} \\ \frac{\partial \theta_z^*}{\partial x} \end{Bmatrix} + [E] \begin{Bmatrix} \theta_y \\ \frac{\partial w_o}{\partial y} \\ \theta_y^* \\ \frac{\partial w_o^*}{\partial y} \\ v_o^* \\ \frac{\partial \theta_z}{\partial y} \\ \frac{\partial \theta_z^*}{\partial y} \end{Bmatrix}$$

where the matrices $[A], [A'], [B], [B'], [D], [D'], [E], [E']$ are the matrices of plate stiffnesses whose elements are already reported in article [25].

3. Analytical solutions

Here the exact solutions of Eqs. (9)–(18) for antisymmetric angle-ply plates are considered. Assuming that the plate is simply supported with SS-2 boundary conditions [27] in such a manner that tangential displacement is admissible, but the normal displacement is not, the following boundary conditions are appropriate:

At edges $x = 0$ and $x = a$;

$$\begin{aligned} &u_o = 0; \quad w_o = 0; \quad \theta_y = 0; \quad \theta_z = 0; \quad M_x = 0; \quad N_{xy} = 0; \\ &u_o^* = 0; \quad w_o^* = 0; \quad \theta_y^* = 0; \quad \theta_z^* = 0; \quad M_x^* = 0; \quad N_{xy}^* = 0 \end{aligned} \tag{19}$$

At edges $y = 0$ and $y = b$;

$$\begin{aligned} &v_o = 0; \quad w_o = 0; \quad \theta_x = 0; \quad \theta_z = 0; \quad M_y = 0; \quad N_{xy} = 0; \\ &v_o^* = 0; \quad w_o^* = 0; \quad \theta_x^* = 0; \quad \theta_z^* = 0; \quad M_y^* = 0; \quad N_{xy}^* = 0 \end{aligned} \tag{20}$$

Following Navier's approach [27–29], the solution to the displacement variables satisfying the above boundary conditions can be expressed in the following forms:

$$\begin{aligned}
u_o &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{o_{mn}} \sin\alpha x \cos\beta y e^{-i\omega t} \\
u_o^* &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{o_{mn}}^* \sin\alpha x \cos\beta y e^{-i\omega t} \\
v_o &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} v_{o_{mn}} \cos\alpha x \sin\beta y e^{-i\omega t} \\
v_o^* &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} v_{o_{mn}}^* \cos\alpha x \sin\beta y e^{-i\omega t} \\
w_o &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{o_{mn}} \sin\alpha x \sin\beta y e^{-i\omega t} \\
w_o^* &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{o_{mn}}^* \sin\alpha x \sin\beta y e^{-i\omega t} \\
\theta_x &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{x_{mn}} \cos\alpha x \sin\beta y e^{-i\omega t} \\
\theta_x^* &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{x_{mn}}^* \cos\alpha x \sin\beta y e^{-i\omega t} \\
\theta_y &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{y_{mn}} \sin\alpha x \cos\beta y e^{-i\omega t} \\
\theta_y^* &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{y_{mn}}^* \sin\alpha x \cos\beta y e^{-i\omega t} \\
\theta_z &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{z_{mn}} \sin\alpha x \sin\beta y e^{-i\omega t} \\
\theta_z^* &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{z_{mn}}^* \sin\alpha x \sin\beta y e^{-i\omega t} \\
p_z^+ &= 0
\end{aligned} \tag{21}$$

where $\alpha = \frac{m\pi}{a}, \beta = \frac{n\pi}{b}$ and ω is the natural frequency of the system. Substituting Eqs. (19)–(21) in to Eq. (9) and collecting the coefficients one obtains

$$([X]_{12\times12}-\lambda[M]_{12\times12})\begin{Bmatrix} u_o \\ v_o \\ w_o \\ \theta_x \\ \theta_y \\ \theta_z \\ u_o^* \\ v_o^* \\ w_o^* \\ \theta_x^* \\ \theta_y^* \\ \theta_z^* \end{Bmatrix}_{12\times1} = \{0\} \tag{22}$$

where $\lambda = \omega^2$ for any fixed values of m and n. The elements of coefficient matrix $[X]$ and mass matrix $[M]$ are already reported in Refs. [25,17] respectively.

4. Numerical results and discussion

In this section, various numerical examples solved are described and discussed for establishing the accuracy of the theory for the free vibration analysis of antisymmetric angle-ply laminated composite and sandwich plates. For all the problems a simply supported plate with SS-2 boundary conditions is considered for the analysis. Results are obtained in closed-form using *Navier's*solution technique by solving the eigenvalue equation. The non-dimensionalized natural frequencies computed for two, four and eight layer antisymmetric angle-ply square laminate with layers of equal thickness are given in Tables 1 and 2.

The orthotropic material properties of individual layers in all the above laminates considered are $E_1/E_2 =$ open, $E_2 = E_3$, $G_{12} = G_{13} = 0.6E_2$, $G_{23} = 0.5E_2$, $\upsilon_{12} = \upsilon_{13} = \upsilon_{23} = 0.25$.

The variation of natural frequencies with respect to side-to-thickness ratio a/h is presented in Table 1. The natural frequencies obtained using the present theory are compared with Reddy's theory. In the case of thick plates (a/h ratios 2, 4, 5 and 10) there is a considerable difference exists between the results computed using the present and the Reddy's theory. The variation of natural frequencies with respect to side-to-thickness ratio a/h for different E_1/E_2 ratio is presented in Table 2. For a four layered thick plate with a/h ratio equal to 2 and E_1/E_2 ratio equal to 3 and 10, the percentage difference in values predicted by present theory are 0.13% and 3.51% lower as compared to Reddy's theory. At higher range of E_1/E_2 ratio equal to 20–40, the percentage difference in values between both the theories is very much higher and Reddy's theory very much over predicts the natural frequency values. For a four layered thick plate with a/h ratio equal to 2 and E_1/E_2 ratio equal to 20, 30 and 40, the percentage difference in values predicted by present theory are 6.08%, 7.99% and 9.70% lower as compared to Reddy's theory. The difference between the models tends to reduce for thin and relatively thin plates. Irrespective of the number of layers the percentage difference in values between the two theories increases with the increase in the degree of anisotropy. As the number of layer increases, the percentage difference in values between the two theories decreases significantly.

The variation of fundamental frequency with respect to the various parameter like the side-to-thickness ratio (a/h), thickness of the core to thickness of the flange (t_c/t_f) and the aspect ratio (a/b) of a five layer sandwich plate with antisymmetric angle-ply face sheets are given in Tables 3 and 4. The following material properties are used for face sheets and the core [23]:

- Face sheets (Graphite-epoxy T300/934)

$E_1 = 19 \times 10^6$ psi (131 GPa)

$E_2 = 1.5 \times 10^6$ psi (10.34 GPa)

$E_2 = E_3$, $G_{12} = 1 \times 10^6$ psi (6.895 GPa)

$G_{13} = 0.90 \times 10^6$ psi (6.205 GPa)

$G_{23} = 1 \times 10^6$ psi (6.895 GPa)

$\upsilon_{12} = 0.22$, $\upsilon_{13} = 0.22$, $\upsilon_{23} = 0.49$

$\rho = 0.057$ lb/in.^{3} (1627 kg/m^3)

Table 1
Non-dimensionalized fundamental frequencies $\overline{\omega} = (\omega b^2/h)\sqrt{\rho/E_2}$ for a simply supported antisymmetric angle-ply square laminated plate

Lamination and number of layers	Source	a/h								
		2	4	5	10	12.5	20	25	50	100
(45°/−45°)$_1$	Present	5.3325	8.8426	10.0350	12.9115	13.4690	14.1705	14.3500	14.6012	14.6668
	HSDT [5][a]	6.2837	9.7593	10.8401	13.2630	13.7040	14.2463	14.3827	14.5723	14.6214
(45°/−45°)$_2$	Present	5.5674	10.0731	11.9465	17.8773	19.4064	21.6229	22.2554	23.1949	23.4499
	HSDT [5][b]	6.1067	10.6507	12.5331	18.3221	19.7621	21.8063	22.3798	23.2236	23.4507
(45°/−45°)$_4$	Present	5.9234	10.7473	12.7523	19.1258	20.7784	23.1829	23.8713	24.8959	25.1741
	HSDT [5][a]	6.2836	10.9905	12.9719	19.2659	20.8884	23.2388	23.9091	24.9046	25.1744

$E_1/E_2 = 40$, $E_2 = E_3$, $G_{12} = G_{13} = 0.6E_2$, $G_{23} = 0.5E_2$, $\upsilon_{12} = \upsilon_{13} = \upsilon_{23} = 0.25$.

[a] Results using this theory are computed independently and are found to be the same as reported in the Ref. [6].

[b] Results using this theory are computed independently for the first time.

Table 2
Non-dimensionalized fundamental frequencies $\varpi = (\omega b^2/h)\sqrt{\rho/E_2}$ for a simply supported antisymmetric angle-ply square laminated plate

Lamination and number of layers	E_1/E_2	Source	a/h					
			2	4	10	20	50	100
(45°/−45°)$_1$	3	Present	4.5312	6.1223	7.1056	7.3001	7.3583	7.3666
		HSDT [5][b]	4.5052	6.0861	7.0739	7.2704	7.3292	7.3373
	10	Present	4.9742	7.2647	8.9893	9.3753	9.4943	9.5123
		HSDT [5][b]	5.1718	7.3469	8.9660	9.3265	9.4377	9.4538
	20	Present	5.1817	8.0490	10.6412	11.2975	11.5074	11.5385
		HSDT [5][b]	5.7094	8.4151	10.7151	11.2772	11.4553	11.4819
	30	Present	5.2771	8.5212	11.8926	12.8422	13.1566	13.2035
		HSDT [5][b]	6.0681	9.1752	12.0971	12.8659	13.1154	13.1521
	40	Present	5.3325	8.8426	12.9115	14.1705	14.6012	14.6668
		HSDT [5][a]	6.2837	9.7593	13.2630	14.2463	14.5723	14.6214
(45°/−45°)$_2$	3	Present	4.6498	6.4597	7.6339	7.8724	7.9442	7.9545
		HSDT [5][b]	4.6546	6.4554	7.6267	7.8649	7.9366	7.9472
	10	Present	5.2061	8.3447	11.4116	12.2294	12.4952	12.5351
		HSDT [5][b]	5.3887	8.5119	11.4674	12.2380	12.4866	12.5238
	20	Present	5.4140	9.3306	14.4735	16.2570	16.8949	16.9927
		HSDT [5][b]	5.7431	9.6855	14.6609	16.3146	16.8964	16.9848
	30	Present	5.5079	9.7966	16.4543	19.2323	20.3134	20.4839
		HSDT [5][b]	5.9481	10.2785	16.7750	19.3499	20.3277	20.4807
	40	Present	5.5674	10.0731	17.8773	21.6229	23.1949	23.4499
		HSDT [5][b]	6.1067	10.6507	18.3221	21.8063	23.2236	23.4507

[a] Results using this theory are computed independently and are found to be the same as reported in the Ref. [6].
[b] Results using this theory are computed independently for the first time.

Table 3
Non-dimensionalized fundamental frequencies $\varpi = (\omega b^2/h)\sqrt{(\rho/E_2)_f}$ for a simply supported antisymmetric angle-ply (45°/−45°/core/45° /−45°) square sandwich plate

t_c/t_f	Source	a/h					
		2	4	10	20	50	100
4	Present	2.6404	4.5712	9.8197	15.0371	19.1695	20.0845
	HSDT [5][a]	3.0986	5.7985	12.0510	16.8312	19.6858	20.2163
10	Present	1.2805	2.1911	5.0653	9.2740	16.2062	19.3098
	HSDT [5][a]	1.6929	3.2171	7.4895	12.6964	18.4604	20.1355
20	Present	0.7538	1.3487	3.2154	6.1552	12.4654	16.7293
	HSDT [5][a]	0.9806	1.8783	4.5392	8.4083	14.9592	18.0073
50	Present	0.6079	1.1836	2.8972	5.5259	10.8499	14.1053
	HSDT [5][a]	0.6473	1.2696	3.1080	5.8904	11.2731	14.3233

[a] Results using this theory are computed independently for the first time.

Table 4
Non-dimensionalized fundamental frequencies $\varpi = (\omega b^2/h)\sqrt{(\rho/E_2)_f}$ for a simply supported antisymmetric angle-ply (45°/−45°/core/45° /−45°) sandwich plate with $a/h = 10$

a/b	Source	t_c/t_f			
		4	10	20	50
1.0	Present	9.8197	5.0653	3.2154	2.8972
	HSDT [5][a]	12.0510	7.4895	4.5392	3.1080
1.5	Present	5.7975	2.9101	1.8354	1.6498
	HSDT [5][a]	7.2503	4.3308	2.5939	1.7706
2.0	Present	4.1579	2.0562	1.2900	1.1557
	HSDT [5][a]	5.2441	3.0627	1.8216	1.2405
2.5	Present	3.2833	1.6054	1.0020	0.8939
	HSDT [5][a]	4.1585	2.3878	1.4122	0.9595
3.0	Present	2.7355	1.3268	0.8241	0.7315
	HSDT [5][a]	3.4698	1.9660	1.1577	0.7849

[a] Results using this theory are computed independently for the first time.

- Core properties (isotropic)

$$E_1 = E_2 = E_3 = 2G = 1000 \text{ psi } (6.90 \times 10^{-3} \text{ GPa})$$

$$G_{12} = G_{13} = G_{23} = 500 \text{ psi } (3.45 \times 10^{-3} \text{ GPa})$$

$$\upsilon_{12} = \upsilon_{13} = \upsilon_{23} = 0$$

$$\rho = 0.3403 \times 10^{-2} \text{ lb/in.}^3 \ (97 \text{ kg/m}^3)$$

The results clearly show that in the case of thick plates for all the parameters considered, there is a considerable difference exists between the results computed using the present theory and Reddy's theory. In the case of a square plate with t_c/t_f ratio equal to 4 and a/h ratio equal to 10, the percentage difference in values predicted by Reddy's theory is 22.72% higher compared to present theory. For a rectangular plate with a/b ratio equal to 2 and t_c/t_f ratio equal to 10, Reddy's theory overestimates the natural frequency by 48.95%. The Reddy's theory very much overestimates the natural frequency values both for square and rectangular plates.

5. Conclusion

Analytical formulations and solutions to the natural frequency analysis of simply supported antisymmetric angle-ply composite and sandwich plates hitherto not reported in the literature based on a higher order refined theory which takes in to account the effects of both transverse shear and transverse normal deformations are presented. The accuracy of the present computational model with 12 degrees of freedom in comparison to other higher order model with five degrees of freedom has been established. It has been concluded that for all the parameters considered Reddy's theory very much over predicts the natural frequency values both for the composite and sandwich plates.

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