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Combinatorial Labelings Of Graphs*

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Abstract

A (p, q)-graph G is said to be a permutation (combination) graph if G admits an assignment of distinct integers $1, 2, 3, ..., p$ to the vertices such that edge values obtained by the number of permutations (combinations) of larger vertex value taken smaller vertex value at a time are distinct. In this paper we obtain a necessary condition for combination graph and study structure of permutation and combination graphs which includes some open problems.

1 Introduction

For all terminology and notation in graph theory we follow Harary [4].

Graph labelings, where the *vertices* and *edges* are assigned *real values* or *subsets of a set* are subject to certain conditions, have often been motivated by their utility to various applied fields and their intrinsic mathematical interest (logical-mathematical). Graph labelings were first introduced in the mid sixties. An enormous body of literature has grown around the subject, especially in the last thirty years or so, and is still getting embellished due to increasing number of application driven concepts [5].

Labeled graphs are becoming an increasingly useful family of *mathematical models* for a broad range of applications. The qualitative labelings of graph elements have inspired research in diverse fields of human enquiry such as *conflict resolution in social psychology*, *electrical circuit theory* and *energy crisis*. Quantitative labelings of graphs have led to quite intricate fields of applications such as *Coding Theory problems*, including the design of good *radar location codes, synch-set codes, missile guidance codes* and *convolution codes with optimal auto-correlation properties*. Labeled graphs have also been applied in determining ambiguities in *X-Ray Crystallographic analysis*, to design *communication network addressing systems*, to determine *optimal circuit layouts and radio-astronomy*, etc.

A graph G consists of a set of vertices and a set of edges. Every edge must join two distinct vertices and no more than one edge may join any vertex pair. If a nonnegative integer $f(v)$ is assigned to each vertex v of G then the vertices of G are said to be labeled (numbered). G is itself a labeled graph if each edge $e = uv$ is given the value

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$f(uv) = f(u) * f(v)$, where $*$ is a binary operation. In the literature one can find that the $*$ is either addition, multiplication, modulo addition or absolute difference, modulo subtraction or symmetric difference. Clearly, in the absence of additional constraints, every graph can be labeled in infinitely many ways. Thus, utilization of labeled graph models requires imposition of additional constraints which characterize the problem being investigated.

"Graph labeling" at its heart, is a strong communication between "number theory" [3] and "structure of graphs" [2,4,5]. In this paper, Combinatorial labelings of graphs are introduced and studied.

Let A be the set of all graphs admitting a labeling f, and B is the set of all graphs admitting a labeling g, then one and only one among the following cases arises.

1 $A \cap B = \phi$.

2 $A \cap B \neq \phi, A \subset B$ and $B \not\subset A$.

3 $A \cap B \neq \phi, A \not\subset B$ and $B \not\subset A$.

4 $A \cap B = A \cup B = A = B$.

It is important to note that the cases (1), (2) and (3) are of interest according to the merit of these cases and in the case (4), it is enough to study the labeling f because either of the labelings yields the identical sets of graphs.

Permutations and combinations play a major role in combinatorial problems. The new labeling introduced in this paper is a logical-mathematical attempt. The contexts in real life which may in future encounter the solutions of problems in this freshly initiated concept are expected to be many.

2 Permutation and Combination Labelings

We begin with the following definition.

DEFINITION 1. A $(p,\ q)$ graph $G = (V,\ E)$ is said to be a *permutation graph* if there exists a bijection $f : V(G) \to \{1, 2, 3, ..., p\}$ such that the induced edge function $g_f : E(G) \to \mathcal{N}$ defined as

$$g_f(uv) = \begin{cases} {}^{f(u)}P_{f(v)} & \text{if } f(u) > f(v) \\ {}^{f(v)}P_{f(u)} & \text{if } f(v) > f(u) \end{cases}$$

is injective, where ${}^{f(u)}P_{f(v)}$ denotes the number of permutations of $f(u)$ things taken $f(v)$ at a time. Such a labeling f is called *permutation labeling* of G.

An example of a permutation graph and a nonpermutation graph are displayed in Figures 1 and 2 respectively.

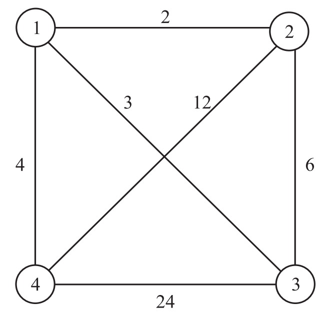

Figure 1. A permutation graph

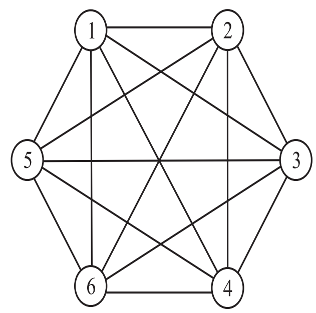

Figure 2. A non-permutation graph (Note that ${}^5P_4 = {}^6P_3$)

THEOREM 1. The complete graph K_n is a permutation graph if and only if $n \leq 5$.

PROOF. By labeling the numbers 1, 2, 3, ..., n to the vertices of K_n; $n \leq 5$ one can easily verify that K_n admits permutation labeling. Conversely, for $n > 5$ we get ${}^6P_3 = {}^5P_4 = 120$. Therefore K_n does not admit permutation labeling as induced edge function is not injective for $n > 5$.

Since the edge values in any permutation labelings are large numbers, investigations of suitable additional constraints to control edge values is a scope for further study. The permutation labelings of all trees up to fifteen vertices are verified. We strongly believe that all trees admit permutation labelings.

DEFINITION 2. A $(p,\ q)$ graph $G = (V,\ E)$ is said to be a *combination graph* if there exists a bijection $f : V(G) \to \{1, 2, 3, ..., p\}$ such that the induced edge function $g_f : E(G) \to \mathcal{N}$ defined as

$$g_f(uv) = \begin{cases} {}^{f(u)}C_{f(v)} & \text{if } f(u) > f(v) \\ \\ {}^{f(v)}C_{f(u)} & \text{if } f(v) > f(u) \end{cases}$$

is injective, where ${}^{f(u)}C_{f(v)}$ is the number of combinations of $f(u)$ things taken $f(v)$ at a time. Such a labeling f is called *combination labeling* of G.

An example of a combination and a noncombination graph are displayed in Figure 3. One can see from Figure 4 that Petersen graph admits combination labeling.

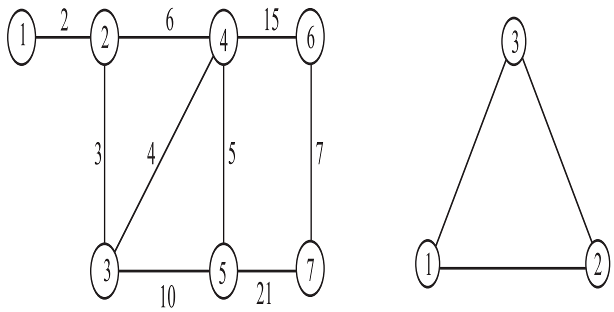

Figure 3. A combination graph and a noncombination graph

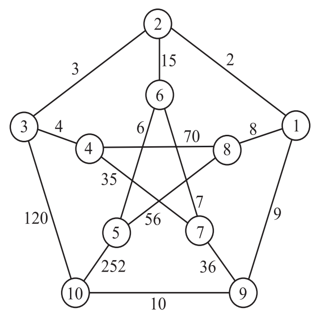

Figure 4. A combination labeling of Petersen graph

THEOREM 2. The cycle C_n admits a combination labeling for all $n > 3$.

PROOF. Denote the vertices of C_n consecutively as $v_1, v_2, v_3, ..., v_n$ such that v_1 is adjacent to v_n and v_i is adjacent to $v_{i+1}, 1 \leq i \leq n-1$. Define a labeling $f : V(C_n) \rightarrow \{1, 2, ..., n\}$ by

$$\begin{cases} f(v_i) & = i \qquad \text{if } 1 \leq i \leq n\text{-}2 \\ f(v_{n-1}) & = n \\ f(v_n) & = n-1 \end{cases}$$

Clearly the above labeling is a combination labeling with edge values set $g_f(E) = \{2, 3, ..., n-1, n, n(n-1)/2\}$. Note that edge values are distinct, otherwise for some r, $0 \leq r \leq n-2$,

$$\frac{n(n-1)}{2} = n - r \Rightarrow n = \frac{3 \pm \sqrt{9-8r}}{2},$$

that is, $9 - 8r > 9$ when $n > 3$. Hence $r < 0$, which is a contradiction. Hence C_n admits a combination labeling for $n > 3$.

For example, a combination labeling of C_6 using above theorem is displayed in Figure 5. The next result gives a necessary condition for a graph to be a combination graph.

THEOREM 3. If a $(p,\ q)$- graph G is a combination graph then

$$4q \leq \begin{cases} p^2 & \text{if } p \text{ is even} \\ p^2 - 1 & \text{if } p \text{ is odd} \end{cases}$$

PROOF. Let f be a combination labeling of a $(p,\ q)$-graph G. Then there exists vertices $v_1,\ v_2,\ ...,v_p$ such that $f(v_1) = 1,\ f(v_2) = 2,\ ...,\ f(v_p) = p$. Since ${}^pC_k = {}^p\, C_{p-k}$, maximum number of distinct values among ${}^pC_1, {}^pC_2, {}^pC_3, ...,\ {}^pC_{p-1}$ is at most $\lfloor p/2 \rfloor$, where $\lfloor x \rfloor$ is the greatest integer of x.

In general the maximum number of distinct values among ${}^{p-r}C_1, {}^{p-r}C_2, {}^{p-r}C_3, ...,$ ${}^{p-r}C_{p-r-1},\ 1 \leq r \leq p-2$ is $\lfloor (p-r)/2 \rfloor$. But the distinct values among ${}^pC_1,\ {}^pC_2,\ ...,$ ${}^pC_{p-1}$ and distinct values among ${}^{p-r}C_1,\ {}^{p-r}C_2,\ ...,\ {}^{p-r}c_{p-r-1},\ 1 \leq r \leq p-2$, taken together need not be distinct. For example, when $p = 6$, we get ${}^6C_1 = {}^4C_2$. Since f is a combination labeling of G, there exists q distinct values on the edges. Therefore, clearly

$$q \leq \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{p-1}{2} \right\rfloor + \cdots + \left\lfloor \frac{p-(p-2)}{2} \right\rfloor.$$

Case 1. Let $p = 2k$. Then from the above equation, we get

$$\begin{aligned} q &\leq \left\lfloor \frac{2k}{2} \right\rfloor + \left\lfloor \frac{2k-1}{2} \right\rfloor + \cdots + \lfloor 1 \rfloor \\ &= k + (k-1) + (k-1) + (k-2) + (k-2) + \cdots + 1 + 1 \\ &= k^2 = \frac{p^2}{4}. \end{aligned}$$

Case *2*. Let $p = 2k+1$. Then from the above equation, we get

$$\begin{aligned} q &\leq \left\lfloor \frac{2k+1}{2} \right\rfloor + \left\lfloor \frac{2k}{2} \right\rfloor + \cdots + \lfloor 1 \rfloor \\ &= k + k + (k-1) + (k-1) + (k-2) + (k-2) + \cdots + 1 + 1 \\ &= k^2 + k = \frac{p^2-1}{4}. \end{aligned}$$

From these two cases, we get

$$4q \leq \begin{cases} p^2 & \text{if } p \text{ is even} \\ p^2 - 1 & \text{if } p \text{ is odd.} \end{cases}$$

COROLLARY 1. The complete graph K_n is a combination graph if and only if $n \leq 2$.

Note that $q = {}^nC_2 = n(n-1)/2$. Therefore $4q = 2n^2 - 2n > n^2$ for $n > 2$. Hence by Theorem 3, for $n > 2$, K_n does not admit combination labeling.

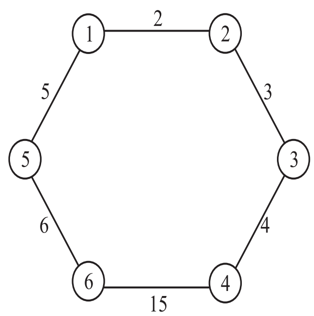

Figure 5. A combination labeling of C_6

By trial and error method one can verify that, the complete bipartite graphs $K_{3,\,3}$ and $K_{4,\,3}$ do not admit combination labeling even though they satisfy necessary condition mentioned in Theorem 3. Hence the converse of Theorem 3 is not true.

In general it is evident that,even though the complete bipartite graph $K_{m,\,n}$ satisfies necessary condition for all m and n, it do not admit combination labeling for all m and n. Thus for what values of m and n, the complete bipartite graph $K_{m,\,n}$ admits combination labeling?, is an interesting question to be tackled.

In this direction the following theorem gives a solution for $m = n = r$.

THEOREM 4. The complete bipartite graph $K_{r,\,r}$ is a combination graph if and only if $r \leq 2$.

PROOF. Clearly $K_{1,\,1}$ and $K_{2,\,2}$ are combination graphs. Suppose $K_{r,\,r}$ is a combination graph for some $r \geq 3$. Let A and B be the sets of labels on the two partite sets. Without loss of generality one can assume that $1 \in A$. Let $A = \{1, x_1, x_2, ..., x_{r-1}\}$ where $x_1 < x_2 < \cdots < x_{r-1}$. As ${}^{x+1}C_1 = {}^{x+1}C_x$, note that,

$$1,\ x \in A \Longrightarrow 1 + x \notin B. \tag{1}$$

Now $1 + x_{r-2} \notin B$. Therefore $1 + x_{r-2} \in A$, which implies $1 + x_{r-2} = x_{r-1}$. Similarly $1 + x_{r-3} \in A$ implies $1 + x_{r-3} = x_{r-2}$. One can easily verify that

$$A = \{1, x_1, x_1 + 1, x_1 + 2, ..., x_1 + r - 2\}.$$

Let $B = \{y_1, y_2, ..., y_r\}$ with $y_1 < y_2 < \cdots < y_r$. Then we must get one and only one of the following.

$$1 \ < \ x_1 \ < \ x_1 + 1 < \ x_1 + 2 \ < \cdots \ < \ x_1 + r - 2 \ < \ y_1 \ < \ y_2 \ < \cdots \ < \ y_r. \tag{2}$$

or

$$1 \ < \ y_1 \ < \ y_2 \ < \ \cdots \ < \ y_r \ < \ x_1 \ < \ x_1 + 1 < \ x_1 + 2 \ < \ \cdots \ < \ x_1 + r - 2. \tag{3}$$

or
There exists $k,\ 0 < k \ < r$, such that

$$\begin{aligned} 1 \ &< \ y_1 < y_2 < \cdots < y_k < x_1 < x_1 + 1 < x_1 + 2 \\ &< \ \cdots < x_1 + r - 2 < y_{k+1} < y_{k+2} < \cdots < y_r. \end{aligned} \tag{4}$$

If $r \geq 3$ then from (2) we get, 1 and x_1+r-2 belong to A. That implies $1+(x_1+r-2) = y_1$ which belong to B, a contradiction to (1). From (3) we have, $y_1+y_r = x_1+1 \in A$, a contradiction to (1). As 1, $(x_1+r-2) \in A$, from (4) we get $1+(x_1+r-2) = y_{k+1} \in B$, a contradiction to (1). Hence $K_{r,\, r}$ is not a combination graph for $r \geq 3$.

THEOREM 5. The wheel graph W_n is not a combination graph for all $n \leq 6$.

PROOF. The number of vertices of $W_n = p = n+1$ and the number of edges is equal to $q = 2n$. Note that $4q = 8n > (n+1)^2$ or $(n+1)^2 - 1$ according as p is even or odd. Since this is possible only for $n \leq 5$, by Theorem 3, W_n is not a combination graph for $n \leq 5$. If $n = 6$ then, one can observe that there are 12 edges in W_6 and we have only 11 distinct values (see Pascal's triangle [3]) 2, 3, 4, 5, 6, 7, 10, 15, 20, 21 and 35 to label the edges which implies W_6 is not a combination graph. Hence W_n is not a combination graph for $n \leq 6$.

Combination labelings of W_7 and W_8 are displayed in Figures 6 and 7.

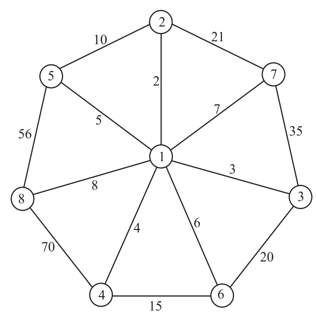

Figure 6. A combination wheel W_7

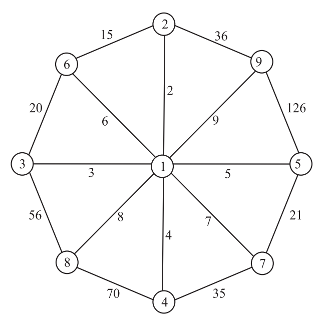

Figure 7. A combination wheel W_8

From the above examples and Pascal's triangle of [3], we strongly believe that the wheel graph W_n is a combination graph for all integers $n \geq 7$, and every tree is a combination graph.

Since combination graphs also demand large numbers on the edges for higher order graphs, we state one suitable additional constraint to control the edge values as $(k,\ d)$-arithmetic labelings to strongly k-indexable labelings [1]. Hence the following definition also gives scope for further study in this area.

DEFINITION 3. A $(p,\ q)$ graph $G=(V,\ E)$ is said to be a *strong k-combination graph* if there exists a bijection $f: V(G) \to \{1,\ 2,\ 3,\ ...,p\}$ such that the induced edge function $g_f: E(G) \to \{k,\ k+1,\ k+2,...,k+q-1\}$ defined by

$$g_f(uv) = \begin{cases} {}^{f(u)}C_{f(v)} & \text{if } f(u) > f(v) \\ {}^{f(v)}C_{f(u)} & \text{if } f(v) > f(u) \end{cases}$$

is also bijective for some positive integer k. Such a labeling f is called *strong k-combination labeling* of G.

For example, paths and stars are strong k-combination graphs. Another example for strong 2-combination graph is displayed in Figure 8.

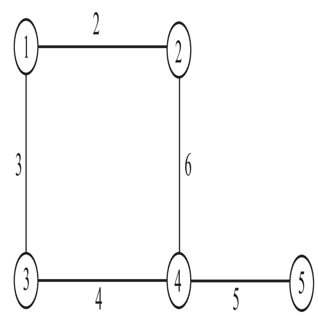

Figure 8. A strong 2-combination graph

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